L^p-Asymptotics of Fourier transform of fractal measures

A Dissertation
submitted in partial fulfilment
of the requirements for the
award of the degree of
Doctor of Philosophy

by

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May 2015

Declaration

I hereby declare that the work reported in this thesis is entirely original and has been carried out by me under the supervision of Professor E. K. Narayanan at the Department of Mathematics, Indian Institute of Science, Bangalore. I further declare that this work has not been the basis for the award of any degree, diploma, fellowship, associateship or similar title of any University or Institution.

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May 2015.

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Acknowledgement

Foremost, I would like to express my sincere gratitude to my research supervisor, Prof. E. K. Narayanan for his guidance and immense support. For his enormous contribution of his time and ideas, I am at loss of words to express my gratefulness. He has always been there to clarify my doubts and guide me. I am highly indebted to him for his unflinching encouragement and infinite patience throughout the course.

I have been immeasurably benefited from various courses that I attended inside and outside the campus and I am grateful to all the instructors. I specially thank Prof. S. Thangavelu, Prof. A. Sitaram and Prof. Malabika Pramanik for their intense teaching and useful discussions. The problems suggested by Prof. Robert Strichartz kindled my interest more in this topic and I sincerely thank him for his valuable remarks. I am also infinitely grateful to Prof. Kaushal Verma and Prof. Gautam Bharali for their valuable time, advices and encouragement on various issues. I owe special acknowledgement to all my inspirational teachers.

My life in IISc is a dream come true. I would like to thank the system administrators, librarians and all the staff members of the Department of Mathematics for the comfortable learning experience and wonderful study atmosphere. I gratefully acknowledge the financial assistance from the Council of Scientific and Industrial Research, Government of India, through its research fellowship for the period of January 2010 - July 2014. I am obliged to Prof Kaushal Verma for

the financial assistance provided during the period August 2014 - July 2015. I also thank the administration of IISc for the travel grant provided to attend a conference in Poland. This work is supported in part by UGC Centre for Advanced Studies.

It would not been such a convivial place to do PhD if not for my friends. Be it near or far, I have been immensely helped and taught by many. I thank each and everyone for their emotional support, helpful discussions and thought provoking interactions. Special thanks to everyone in the campus who took care of me with warm hospitality throughout this course of time. I am always indebted to my family members who were supportive, understanding and not just enduring this process with me but also helped me come out of all kinds of frustrations.

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Introduction

One of the basic questions in harmonic analysis is to study the decay properties of the Fourier transform of measures or distributions supported on thin sets in $\mathbb{R}^n$. When the support is a smooth enough manifold, an almost complete picture is available. One of the early results in this direction is the following: *Let $f \in C_c^\infty(\mathbb{R}^n)$ and $d\sigma$ be the surface measure on the sphere $S^{n-1} \subset \mathbb{R}^n$. Then*

$$|\widehat{fd\sigma}(\xi)| \leq \ C\ (1 + |\xi|)^{-\frac{n-1}{2}}.$$

It follows that $\widehat{fd\sigma} \in L^p(\mathbb{R}^n)$ for all $p > \frac{2n}{n-1}$. This result can be extended to compactly supported measures on $(n-1)$-dimensional manifolds with appropriate assumptions on the curvature. On the other hand, the results in [2] show that $\widehat{fd\sigma} \notin L^p(\mathbb{R}^n)$ for $1 \leq p \leq \frac{2n}{n-1}$. Similar results are known for measures supported in lower dimensional manifolds in $\mathbb{R}^n$ under appropriate curvature conditions (See page 347-351 in [40]). However, the picture for fractal measures is far from complete. This thesis is a contribution to the study of L^p-integrability and L^p-asymptotic properties of the Fourier transform of measures supported in fractals of dimension $0 < \alpha < n$ for $1 \leq p \leq 2n/\alpha$.

In the first chapter we recall several notions of dimensions (Hausdorff dimension, Packing dimension, etc.) and various geometric properties of fractal sets. Let $0 < \alpha < n$ and $\mathcal{H}_\alpha$ denote the α-dimensional Hausdorff measure. Recall from [41] that a set E is said to be quasi α-regular if for all $0 < r \leq 1$, there exists a constant a such that $ar^\alpha \leq \mathcal{H}_\alpha(E \cap B_r(x))$ for all x. We discuss the relation between quasi α-regular sets and sets of finite α-packing measure ($0 < \alpha < n$) in Chapter 2.

In [1] and [2], the authors related the integrability of the functions and the integer dimension of the support of its Fourier transform. In [2], it was proved that, if $f \in L^p(\mathbb{R}^n)$ such that $supp$ $\widehat{f}$ is carried by a d-dimensional C^1-manifold, then $f \equiv 0$, if $1 \leq p \leq \frac{2n}{d}$. We extend this result by relating the integrability of the function and the fractal dimension of the support of its Fourier transform by proving the following:

Theorem A[37]: *Let $f \in L^p(\mathbb{R}^n)$ be such that $\widehat{f}$ is supported in a set $E \subset \mathbb{R}^n$. Suppose E is a set of finite α-packing measure, $0 < \alpha < n$. Then f is identically zero, provided $p \leq 2n/\alpha$.*

Using the example constructed by Salem in $\mathbb{R}$ (See page 267 in [8]), we show that Theorem A is sharp.

Inspired by results in [41], we look for quantitative estimates for Fourier transform of fractal measures. Let E be a compact set of finite α-packing measure and $\mu = \mathcal{P}^\alpha|_E$. In Chapter 3, we obtain quantitative versions of Theorem A by obtaining lower and upper bounds for the following:

$$\limsup_{L\to\infty} \frac{1}{L^k} \int_{|\xi|\leq L} |\widehat{fd\mu}(\xi)|^p d\xi,$$

where k depends on $\alpha,\ p$ and n.

If μ is a compactly supported locally uniformly α-dimensional measure, that is, $\mu(B_r(x)) \leq ar^\alpha$ for all $0 < r \leq 1$ and some non-zero finite constants a, then in [41], Strichartz proved that there exists constant C_1 independent of f such that

$$\|f\|_{L^2(d\mu)} \geq C_1 \limsup_{L\to\infty} \frac{1}{L^{n-\alpha}} \int_{|\xi|\leq L} |\widehat{fd\mu}(\xi)|^2 d\xi. \tag{0.0.0.1}$$

In addition, if μ is supported in a quasi α-regular set, then there exists a non-zero constant independent of f such that

$$\|f\|_{L^2(d\mu)} \leq C_1 \liminf_{L\to\infty} \frac{1}{L^{n-\alpha}} \int_{|\xi|\leq L} |\widehat{fd\mu}(\xi)|^2 d\xi. \tag{0.0.0.2}$$

The authors in [21] and [22] have generalized (0.0.0.1) for a general class of measures. Using Holder's inequality, we note from (0.0.0.2) that if $f \in L^2(d\mu)$, where μ is a locally uniformly

α-dimensional measure, then for $1 \leq p \leq 2$,

$$\|f\|_{L^2(d\mu)} \leq C \limsup_{L\to\infty} \frac{1}{L^{n-\alpha p/2}} \int_{|\xi|\leq L} |\widehat{fd\mu}(\xi)|^p d\xi. \tag{0.0.0.3}$$

First we consider $2 \leq p < \frac{2n}{\alpha}$. The above results hold for locally uniformly α-dimensional measures. But if E is a set of finite α-packing measure, then $\mu = \mathcal{P}^\alpha|_E$ need not be locally uniformly α-dimensional measure. We first prove an analogue result of (0.0.0.3) for $\mu = \mathcal{P}^\alpha|_E$, where E is of finite α-packing measure:

Theorem B: *Let $f \in L^2(d\mu)$ be a positive function where $\mu = \mathcal{P}^\alpha|_E$ and E is a compact set of finite α-packing measure. Then for $2 \leq p < 2n/\alpha$,*

$$\int_{\mathbb{R}^n} |f(x)|^2 d\mu(x) \leq \ C \liminf_{L\to\infty} \Big(\frac{1}{L^{n-\alpha p/2}} \int_{|\xi|\leq L} \ |\widehat{fd\mu}(\xi)|^p d\xi \Big)^{2/p}.$$

In [1], the authors proved the following:

Theorem 1(Agmon $\&$ **Hormander):** *Let u be a tempered distribution such that $\widehat{u} \in L^2_{loc}(\mathbb{R}^n)$ and*

$$\limsup_{L\to\infty} \frac{1}{L^k} \int_{|\xi|\leq L} |\widehat{u}(\xi)|^2 d\xi < \infty.$$

If the restriction of u to an open subset X of $\mathbb{R}^n$ is supported by a C^1 submanifold M of codimension k, then it is an L^2-density $u_0 dS$ on M and

$$\int_M |u_0|^2 dS \leq C \limsup_{L\to\infty} \frac{1}{L^k} \int_{|\xi|\leq L} |\widehat{u}(\xi)|^2 d\xi,$$

where C only depends on n.

We prove an analogue of the above theorem for fractional dimensional sets.

Theorem C: *Let u be a tempered distribution supported in a set E of finite α-packing measure such that for $2 \leq p < 2n/\alpha$,*

$$\limsup_{L\to\infty} \frac{1}{L^{n-\frac{\alpha p}{2}}} \int_{|\xi|\leq L} |\widehat{u}(\xi)|^p d\xi < \infty.$$

Then u is an L^2 density $u_0 \, d\mathcal{P}^\alpha$ on E and

$$\Big(\int_E |u_0|^2 d\mathcal{P}^\alpha \Big)^{p/2} \leq \ C \limsup_{L\to\infty} \frac{1}{L^{n-\frac{\alpha p}{2}}} \int_{|\xi|\leq L} |\widehat{u}(\xi)|^p d\xi < \infty.$$

In a different direction, we consider the result of Hudson and Leckband in [17]. For $0 < \alpha < 1$, the authors defined α-coherent set in $\mathbb{R}$. A set $E \subset \mathbb{R}^n$ of finite α-dimensional Hausdorff measure is called α-coherent if for all x,

$$\limsup_{\epsilon \to 0} |E_x^0(\epsilon)| \epsilon^{\alpha - n} \le C_E \mathcal{H}_\alpha(E_x^0),$$

where $E_x^0 = \{y \in E : y \le x \text{ and } 2^{-\alpha} \le \limsup_{\delta \to 0} \frac{\mathcal{H}_\alpha(E \cap (y-\delta, y+\delta))}{\delta^\alpha} \le 1\}$ and $E_x^0(\epsilon)$ denotes the ϵ-distance set of E_x^0.

Theorem 2(Hudson & Leckband): *Let $E \subset \mathbb{R}$ be either an α-coherent set or a quasi α-regular set of finite α-dimensional Hausdorff measure, for $0 < \alpha < 1$, and $f \in L^1(d\mu)$, where $\mu = \mathcal{H}_\alpha|_E$. Then there is a constant C independent of f such that*

$$\int_E \frac{|f(x)|}{\mu(E_x)} d\mu(x) \le C \liminf_{L \to \infty} \frac{1}{L^{1-\alpha}} \int_{-L}^{L} |\widehat{f d\mu}(\xi)| d\xi.$$

The authors in [17] also proved a Hardy type inequality for discrete measures which we state below. Let $\|u\|_{B^p.a.p}^p = \lim\ L^{-1} \int_{-L}^{L} |u(x)|^p dx$. The authors in [17] proved the following:

Theorem 3(Hudson & Leckband): *Let c_k be a sequence of complex numbers and a_k be a sequence of real numbers not necessarily increasing. Let $f d\mu_0$ be the zero-dimensional measure $f(x) = \sum_1^\infty c_k \delta(x - a_k)$ and let $1 < p \le 2$. Assume that $u(x) = \widehat{f d\mu_0}(x)$. Then if c_k^* denote the nonincreasing rearrangement of the sequence $|c_k|$,*

$$\sum_1^\infty \frac{|c_k|^p}{k^{2-p}} \le \sum_1^\infty \frac{|c_k^*|^p}{k^{2-p}} \le C\ \|u\|_{B^p.a.p}^p.$$

Using the packing measure and finding a continuous analogue of the arguments in [17], we extend Theorem 2 to $\mathbb{R}^n$ and generalize Theorem 3 to any α, $0 < \alpha < n$ and $n \ge 1$ with a slight modification in the hypothesis. Let $E_x = E \cap (-\infty, x_1] \times ... \times (-\infty, x_n]$ for $x = (x_1, ...x_n) \in \mathbb{R}^n$:

Theorem D: *Let $0 < \alpha < n$. Let E be a set of finite α-packing measure. We denote $\mu = \mathcal{P}^\alpha|_E$, where $\mathcal{P}^\alpha$ is the packing measure. Let $f \in L^p(d\mu)$ $(1 \le p \le 2)$ be a positive function. Then there exists a constant C independent of f such that*

$$\int \frac{|f(x)|^p}{(\mu(E_x))^{2-p}} d\mu(x) \le C \liminf_{L \to \infty} \frac{1}{L^{n-\alpha}} \int_{B_L(0)} |\widehat{f d\mu}(\xi)|^p d\xi.$$

The following is the key ingredient of the proof: Let ϕ be a radial Schwartz function on $\mathbb{R}^n$ such that $\widehat{\phi}$ is supported in the unit ball and $\widehat{\phi}(0) = 1$. Let $\phi_L(x) = \phi(Lx)$. Then $\widehat{\phi_L}(x) = L^{-n}\widehat{\phi}(\frac{x}{L})$. We approximate $fd\mu$ using ϕ_L on a finer decomposition $\{S_k\}_k$ of $1/L$-distance set $E(1/L)$ of E for large L. We then prove the result for $p = 1$. The result for $p = 2$ follows from Plancherel theorem. Then using interpolation we prove the result for $1 < p < 2$.

Theorem C and Theorem D can also be proved if the assumptions on E in the hypothesis is changed to quasi α-regular set of finite α-Hausdorff measure with $\mu = \mathcal{H}_\alpha|_E$.

As an application, we use Theorem A to prove some L^p-Wiener-Tauberian theorems. N. Wiener[44] characterized the cyclic vectors (with respect to translations) in $L^p(\mathbb{R})$, for $p = 1, 2$, in terms of the zero set of the Fourier transform. He conjectured that a similar characterization should be true for $1 < p < 2$(See page 93 in [44]). Segal [36], Edwards [9], Rosenblatt and Shuman[32] have disproved the conjecture. Lev and Olevskii in [23] recently proved that for any $1 < p < 2$ one can find two functions in $L^1(\mathbb{R}) \cap C_0(\mathbb{R})$, such that one is cyclic in $L^p(\mathbb{R})$ and the other is not, but their Fourier transforms have the same (compact) set of zeros. This disproves Wiener's conjecture. As is well known, there are no complete answers to L^p-Weiner-Tauberian theorems when $p \neq 1, 2$. See pages 234-236 in [8] for initial results. The problem has been studied by , Pollard [30], Beurling [6], Herz [15], Newman [29], Kinukawa [20], Rawat and Sitaram [31].

In [6], A. Beurling proved that if the Hausdorff dimension of the closed set where the Fourier transform of f vanishes is α for $0 \leq \alpha \leq 1$, then the space of finite linear combinations of translates of f is dense in $L^p(\mathbb{R})$ for $2/(2-\alpha) < p$. Now using our result we prove a similar result (including the end points for the range) on $\mathbb{R}^n$ where sets of Hausdorff dimension is replaced with the sets of finite packing α-measure.

C. S Herz studied some versions of L^p- Wiener Tauberian theorems and gave alternative sufficient conditions for the translates of $f \in L^1 \cap L^p(\mathbb{R}^n)$ to span $L^p(\mathbb{R}^n)$ (See [15]). With an additional hypothesis on the zero sets of Fourier transform of f, we improve his result.

In [31], Rawat and Sitaram initiated the study of L^p-versions of the Wiener Tauberian theo-

rem under the action of motion group $M(n)$ on $\mathbb{R}^n$. We shall show that some of the results proved in [31] can be improved using our result. Finally we take up L^p-Wiener Tauberian theorem on the Euclidean motion group $M(2)$.

The plan of the thesis is as follows. In the next chapter, we set up notation and recall definitions and results that are needed for our results. We start Chapter 2 by studying the relation between quasi α-regular sets and sets of finite α-packing measure and we prove Theorem A and its sharpness. In Chapter 3 we prove quantitative statements of Theorem A. In Chapter 4 we apply the Theorem A to prove Wiener-Tauberian type theorems on $\mathbb{R}^n$ and $M(2)$. We end the thesis with a few open problems to be studied in the future.

Chapter 1

Preliminaries

In this chapter, we recall some definitions and some results from [7], [12] and [24] which will be used throughout this thesis.

1.1 Fractal Geometry

Let X be a metric space, $\mathcal{F}$ a family of subsets of X such that for every $\delta > 0$ there are $E_1, E_2, ... \in \mathcal{F}$ such that diameter of E_k is less than or equal to δ for all k and $X = \cup_k E_k$. For $0 < \delta \leq \infty$ and $A \subset X$, we define **s-dimensional Hausdorff measure** as

$$\mathcal{H}_s(A) = \lim_{\delta \to 0} H^s_\delta(A),$$

where

$$\mathcal{H}^s_\delta(A) = \inf \left\{ \sum_{i=1}^{\infty} d(E_i)^s : A \subset \underset{i}{\cup} E_i, d(E_i) \leq \delta, E_i \in \mathcal{F} \right\}$$

and $d(E)$ denotes the diameter of the set E. If the family $\mathcal{F}$ of subsets of X is replaced by the family of closed (or open) balls, then the resulting measure denoted by $\mathcal{S}_s$ is called **s-dimensional spherical measure**, that is,

$$\mathcal{S}_s(A) = \lim_{\delta \to 0} S^s_\delta(A),$$

where

$$\mathcal{S}_\delta^s(A) = \inf \{\sum_{i=1}^{\infty} r_i^s : A \subset \underset{i\in\mathbb{N}}{\cup} B_{r_i}(x_i), r_i \leq \delta\}.$$

Remark 1.1.1. *Spherical and Hausdorff measures are related by the inequalities*

$$\mathcal{H}_t(A) \leq \mathcal{S}_t(A) \leq 2^t \mathcal{H}_t(A).$$

Hence throughout this thesis, we use $\mathcal{F}$ as the family of closed (or open) balls in the definition of Hausdorff measure.

The **Hausdorff dimension** of a set A is given by

$$\begin{aligned}\dim_H(A) &= \sup\{s : \mathcal{H}_s(A) > 0\} = \sup\{s : \mathcal{H}_s(A) = \infty\} \\ &= \inf\{t : \mathcal{H}_t(A) < \infty\} = \inf\{t : \mathcal{H}_t(A) = 0\}.\end{aligned}$$

For a non-empty subset A of $\mathbb{R}^n$, let $A(\epsilon) = \{x \in \mathbb{R}^n \; : \inf_{y\in A} |x-y| < \epsilon\}$ denote the closed ϵ-neighborhood of A. Some authors call $A(\epsilon)$, the **ϵ-parallel set of** A or **ϵ-distance set of** A. Let E be a non-empty bounded subset of $\mathbb{R}^n$. The **ϵ-covering number** of E denoted by $N(E,\epsilon)$, is the smallest number of open balls of radius ϵ needed to cover E. The **upper** and **lower Minkowski dimensions of** E are defined by

$$\overline{\dim}_M(E) = \inf\{s : \limsup_{\epsilon\downarrow 0} N(E,\epsilon)\epsilon^s = 0\}$$

and

$$\underline{\dim}_M(E) = \inf\{s : \liminf_{\epsilon\downarrow 0} N(E,\epsilon)\epsilon^s = 0\}$$

respectively. Similar to the Hausdorff dimension, the upper and lower Minkowski dimensions are given by

$$\begin{aligned}\overline{\dim}_M(E) &= \sup\{s : \limsup_{\epsilon\downarrow 0} N(E,\epsilon)\epsilon^s > 0\} = \sup\{s : \limsup_{\epsilon\downarrow 0} N(E,\epsilon)\epsilon^s = \infty\} \\ &= \inf\{t : \limsup_{\epsilon\downarrow 0} N(E,\epsilon)\epsilon^s < \infty\} = \inf\{t : \limsup_{\epsilon\downarrow 0} N(E,\epsilon)\epsilon^s = 0\}. \\ \underline{\dim}_M(E) &= \sup\{s : \liminf_{\epsilon\downarrow 0} N(E,\epsilon)\epsilon^s > 0\} = \sup\{s : \liminf_{\epsilon\downarrow 0} N(E,\epsilon)\epsilon^s = \infty\} \\ &= \inf\{t : \liminf_{\epsilon\downarrow 0} N(E,\epsilon)\epsilon^s < \infty\} = \inf\{t : \liminf_{\epsilon\downarrow 0} N(E,\epsilon)\epsilon^s = 0\}.\end{aligned}$$

The **upper and lower Minkowski α-contents** of set E are defined by

$$\mathcal{M}^{*\alpha}(E) = \limsup_{\delta\to 0} (2\delta)^{\alpha-n}|E(\delta)|,$$

$$\mathcal{M}_*^\alpha(E) = \liminf_{\delta \to 0} \, (2\delta)^{\alpha - n} |E(\delta)|,$$

where $|E(\delta)|$ denotes the n-dimensional Lebesgue measure of the δ-distance set of E. Then the upper and lower Minkowski dimensions of E are given by

$$\overline{\dim}_M(E) = \inf \{s : \mathcal{M}^{*s}(E) = 0\} = \sup \{s : \mathcal{M}^{*s}(E) > 0\},$$

$$\underline{\dim}_M(E) = \inf \{s : \mathcal{M}_*^s(E) = 0\} = \sup \{s : \mathcal{M}_*^s(E) > 0\}.$$

The **ϵ-packing number** of E denoted by $P(E, \epsilon)$ is the largest number of **disjoint** open balls of radius ϵ with centres in E. The **ϵ-packing** of E is any collection of disjoint balls $\{B_{r_k}(x_k)\}_k$ with centres $x_k \in E$ and radii satisfying $0 < r_k \leq \epsilon/2$. Let $0 \leq s < \infty$. For $0 < \epsilon < 1$ and $A \subset \mathbb{R}^n$, put

$$P_\epsilon^s(A) = \sup \{\sum_k (2r_k)^s\},$$

where the supremum is taken over all permissible ϵ-packings, $\{B_{r_k}(x_k)\}_k$ of A. Then $P_\epsilon^s(A)$ is non-decreasing with respect to ϵ and we set the **packing pre measure**, P_0^s as

$$P_0^s(A) = \lim_{\epsilon \downarrow 0} P_\epsilon^s(A).$$

We have $P_0^s(\emptyset) = 0$, P_0^s is monotonic and finitely subadditive, but not countably sub-additive. The **s-dimensional packing measure** of A denoted by $\mathcal{P}^s(A)$ is defined as

$$\mathcal{P}^s(A) = \inf \, \Big\{ \sum_{i=1}^{\infty} P_0^s(A_i) \; : A \subset \bigcup_{i=1}^{\infty} A_i \Big\}.$$

where infimum is taken over all countable coverings $\{A_k\}_k$ of A.

Recall that μ is called a Borel regular measure on X, if all Borel sets are μ-measurable and for every $A \subset X$, there is a Borel set $B \subset X$ such that $A \subset B$ and $\mu(A) = \mu(B)$. μ is a Radon measure if all Borel sets are μ-measurable and

1. $\mu(K) < \infty$ for compact sets $K \subset X$,
2. $\mu(V) = \sup \{\mu(K) \; : K \subset V \text{is compact}\}$ for open sets $V \subset X$,
3. $\mu(A) = \inf \{\mu(V) \; : A \subset V, \; V \text{ is open}\}$ for $A \subset X$.

Remark 1.1.2. *1. Hausdorff measure is a Borel regular measure. Moreover, if E is a set of finite α-dimensional Hausdorff measure, then μ, the restriction of α-dimensional Hausdorff measure to E is a Radon measure.*

2. $\mathcal{P}^s$ is Borel regular. Similar to Hausdorff measure, if $\mathcal{P}^s(A) < \infty$, then $\nu = \mathcal{P}^s|_A$ is a Radon measure.

(See Theorem 3.11 in [7] for the proof.)

Lemma 1.1.3. *Fix $\epsilon > 0$. Let A be a non-empty bounded subset of $\mathbb{R}^n$ and $|A(\epsilon)|$ denote the Lebesgue measure of $A(\epsilon)$, where A is a non-empty bounded subset of $\mathbb{R}^n$. Then,*

1. $N(A, 2\epsilon) \leq P(A, \epsilon) \leq N(A, \epsilon/2)$.
2. $\Omega_n P(A, \epsilon)\epsilon^n \leq |A(\epsilon)| \leq \Omega_n N(A, \epsilon)(2\epsilon)^n$,
   where Ω_n denotes the volume of the unit ball in $\mathbb{R}^n$.
3. *For* $0 \leq s < \infty,\ P(A, \epsilon/2)\epsilon^s \leq P^s_\epsilon(A)$.

(See pages 78-79 in [24].)

The **lower** and **upper packing dimension** of any subset A of $\mathbb{R}^n$ are defined respectively as

$$\begin{aligned} \underline{\dim}_P(A) &= \inf\{\sup_i \underline{\dim}_M(A_i) : A \subset \cup_{i=1}^{\infty} A_i,\ A_i \text{ is bounded } \forall i\}, \\ \overline{\dim}_P(A) &= \inf\{\sup_i \overline{\dim}_M(A_i) : A \subset \cup_{i=1}^{\infty} A_i,\ A_i \text{ is bounded } \forall i\}. \end{aligned}$$

For any $A \subset \mathbb{R}^n$,

$$\begin{aligned} \overline{\dim}_P(A) &= \sup\{s : \mathcal{P}^s(A) > 0\} = \sup\{s : \mathcal{P}^s(A) = \infty\} \\ &= \inf\{t : \mathcal{P}^t(A) < \infty\} = \inf\{t : \mathcal{P}^t(A) = 0\}. \end{aligned} \tag{1.1.0.1}$$

From the definitions, **the relation between all the three dimensions** is given by the following: For any set $A \subset \mathbb{R}^n$

$$\dim_H(A) \leq \underline{\dim}_P(A) \leq \underline{\dim}_M(A) \tag{1.1.0.2}$$

and

$$\underline{\dim}_P(A) \leq \overline{\dim}_P(A) \leq \overline{\dim}_M(A) \leq n. \tag{1.1.0.3}$$

All these inequalities can be strict.

Example 1.1.4. *Let E be a symmetrical perfect set in $[0,1]$:*

$$E = \bigcap_n E_n,$$

where E_n is the union of 2^n non-overlapping intervals of length a_n, each of them containing two intervals of E_{n+1}. The sequence (a_n) satisfies $a_0 = 1$, $2a_{n+1} < a_n$. Then Tricot in [43] proved that

$$\begin{aligned} \dim_H(E) &= \liminf_n \frac{n \ln 2}{-\ln a_n}, \\ \overline{\dim}_P(E) &= \limsup_n \frac{n \ln 2}{-\ln a_n}. \end{aligned}$$

See pages 72-73 in [43] for more examples that prove the inequalities in (1.1.0.2) and (1.1.0.3) are strict.

Let $\alpha < n$. A set $E \subset \mathbb{R}^n$ is said to be **Ahlfors-David regular α-set** or **α-regular** if there exists non-zero positive finite real numbers a, b such that

$$0 < ar^\alpha \leq \mathcal{H}_\alpha(E \cap B_r(x)) \leq br^\alpha < \infty$$

for all $x \in E$ and $0 < r \leq 1$.

In 1991, A. Salli [35] proved that upper Minkowski dimension of a non-empty bounded α-regular set is α.(See page 80 in [24] also for the proof of the following theorem.)

Theorem 1.1.5. *Let A be a non-empty bounded subset of $\mathbb{R}^n$. Suppose there exists a Borel measure μ on $\mathbb{R}^n$ and positive numbers $a,\ b,\ r_0$ and s such that $0 < \mu(A) \leq \mu(\mathbb{R}^n) < \infty$ and*

$$0 < ar^s \leq \mu(B_r(x)) \leq br^s < \infty \textit{ for } x \in A,\ 0 < r \leq r_0.$$

Then $\dim_H(A) = \underline{\dim}_M(A) = \overline{\dim}_M(A) = s$. Hence $\dim_H(A) = \underline{\dim}_P(A) = \overline{\dim}_P(A) = \underline{\dim}_M(A) = \overline{\dim}_M(A) = s$.

Definition 1.1.6. *A **similitude** S is a map $S : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$S(x) = sR(x) + b,\ x \in \mathbb{R}^n$$

for some isometry R, $b \in \mathbb{R}^n$ and $0 < s < 1$. The number s is called contraction ratio or dilation factor of S. Let $\mathcal{S} = \{S_1, ...S_m\},\ m \geq 2$ be a collection of finite set of similitudes with dilation

*factors $s_1, ..., s_m$ (so that $S_j = s_j R_j + b_j$ where R_j denotes an isometry and $b_j \in \mathbb{R}^n$). We say that a non-empty compact set K is **invariant** under $\mathcal{S}$ if*

$$K = \cup_{j=1}^{m} S_j K.$$

*$\mathcal{S}$ satisfies the **open set condition** if there is a non-empty open set O such that $\cup_{j=1}^{m} S_j(O) \subset O$ and $S_j(O) \cap S_k(O) = \emptyset$ for $j \neq k$. We call the invariant set K under $\mathcal{S}$ to be **self-similar** if with $\alpha = \dim_H(K)$,*

$$\mathcal{H}_\alpha(S_{j_1}(K) \cap S_{j_2}(K)) = 0 \ \textit{for}\ j_1 \neq j_2.$$

Theorem 1.1.7. *If $\mathcal{S}$ satisfies the open set condition, then the invariant set K is self-similar and $0 < \mathcal{H}_\alpha(K) < \infty$, where $\alpha = \dim_H(K)$. Moreover, α is the unique number for which*

$$\sum_{j=1}^{m} s_j^\alpha = 1.$$

Additionally, if O is the open set asserted to exist by the open set condition such that it contains a ball of radius c_1 and it is contained in a ball of radius c_2,

$$\frac{s^\alpha}{diam(K)^\alpha} \leq \frac{\mathcal{H}_\alpha(E \cap B_r(x))}{r^\alpha} \leq \frac{(1+2c_2)^n}{s^n c_1^n},\ \forall\, 0 < r \leq 1,$$

where $diam(K)$ denotes the diameter of K and $s = \min_{j=1}^{m}\{s_j\}$. That is, K is an α-regular set.

(See page 67 in [24] and [18] for proof.)

Remark 1.1.8. *If $m = 2$, $S_1(x) = x/3$, $S_2(x) = x/3 + 2/3$ for $x \in [0, 1]$ in Theorem 1.1.7, then the Cantor set K is invariant under $\mathcal{S} = \{S_1, S_2\}$. The Hausdorff dimension of K is $\ln 2/\ln 3$ and it is self-similar. Hence it is $\ln 2/\ln 3$-regular set.*

If ν is a measure, the α**-upper density of** ν at x, $\overline{D^\alpha}(\nu, x)$ is defined as

$$\overline{D^\alpha}(\nu, x) \ = \ \limsup_{r\to 0}\, (2r)^{-\alpha}\nu(B_r(x)),$$

where $B_r(x)$ is the ball of radius r with centre x. Similarly α**-lower density** of ν at x, $\underline{D^\alpha}(\nu, x)$ is defined using $\liminf$.

In [41], Strichartz defined the following:

- A set $E \subset \mathbb{R}^n$ is said to be **regular**, if $\underline{D^\alpha}(\mu_\alpha, x) = \overline{D^\alpha}(\mu_\alpha, x) = 1$ for $\mathcal{H}_\alpha$-almost all $x \in E$ where $\mu_\alpha = \mathcal{H}_\alpha|_E$.

- A set $E \subset \mathbb{R}^n$ is called **quasi α-regular** if there exists a non-zero finite constant a such that $a \leq \underline{D^\alpha}(\mu_\alpha, x)$ for $\mathcal{H}_\alpha$-almost all $x \in E$.

- A set $E \subset \mathbb{R}^n$ is said to be **locally uniformly α-dimensional** if there exists a non-zero finite constant b such that $\mathcal{H}_\alpha(E \cap B_r(x)) \leq br^\alpha$ for all $x \in E$ and for all $0 < r \leq 1$.

- A measure μ is called locally uniformly α dimensional if there exists a non-zero finite constant λ such that for all $\delta \leq 1$,
$$\mu(B_\delta(x)) \leq \lambda \delta^\alpha. \tag{1.1.0.4}$$
for μ-almost every $x \in \mathbb{R}^n$. Note that E is locally uniformly α-dimensional if and only if $\mathcal{H}_\alpha|_E$ is locally uniformly α-dimensional.

A powerful theorem of Besicovitch [5] shows that every Borel set of infinite $\mathcal{H}_s$ measure contains subsets of arbitrary finite $\mathcal{H}_s$ measure that are locally uniformly s-dimensional (See page 163 in [41] and page 67 in [12]).

Remark 1.1.9. *Clearly, Ahlfors-David α-regular sets are quasi α-regular sets. Also bounded self-similar sets K with self-similar dimension α, are locally uniformly α-dimensional and quasi α-regular. (See Theorem 5.8 in page 179 in [41] for proof.)*

The following lemma gives the relation between the Hausdorff measure and the packing measure of a set:

Lemma 1.1.10. *Let $A \subset \mathbb{R}^n$ be any set.*

1. $\mathcal{H}_s(A) \leq \mathcal{P}^s(A)$.

2. *Let $\mathcal{P}^s(A) < \infty$. $\mathcal{P}^s(A) = \mathcal{H}_s(A)$ if and only if $\underline{D^s}(\nu, x) = \overline{D^s}(\nu, x) = 1$ for $\mathcal{P}^s$-almost all $x \in A$, where ν denotes the Hausdorff measure $\mathcal{H}_s$ restricted to A.*

3. *Let $\mathcal{H}_s(A) < \infty$ and ν denote the Hausdorff measure $\mathcal{H}_s$ restricted to A. If $\underline{D^s}(\nu, x) > 0$ for $\mathcal{P}^s$-almost all $x \in A$, then $\dim_H(A) = \overline{\dim}_P(A)$.*

(See pages 84, 96 and 98 in [24] for the proof.)

The local properties of sets with finite Hausdorff and packing measures can be studied with the help of the following lemma:

Lemma 1.1.11. *Suppose $\mathcal{H}_\alpha(B) < \infty$, for $0 \leq \alpha < n$. Let $\mu = \mathcal{H}_\alpha|_B$. Then*

1. *$2^{-\alpha} \leq \overline{D^\alpha}(\mu, x) \leq 1$ for $\mathcal{H}_\alpha$ almost all $x \in B$.*
2. *$\overline{D^\alpha}(\mu, x) = 0$ for $\mathcal{H}_\alpha$-almost all $x \notin B$.*

Suppose $\mathcal{P}^\alpha(B) < \infty$, for $0 \leq \alpha < n$. Then $\underline{D}^\alpha(\mathcal{P}^\alpha|_B, x) = 1$ for $\mathcal{P}^\alpha$ almost all $x \in B$. (See pages 89-95 in [24] for the proof.)

Information on upper densities of a Radon measure μ can be used to compare μ with Hausdorff measures and similarly information on lower densities of μ can be used to compare μ with packing measure:

Lemma 1.1.12. *Let μ be a Radon measure, $B \subset \mathbb{R}^n$ and $0 < \lambda < \infty$.*

1. *If $\overline{D^\alpha}(\mu, x) \leq \lambda$ for $x \in B$, then $\mu(B) \leq 2^\alpha \lambda \mathcal{H}_\alpha(B)$.*
2. *If $\overline{D^\alpha}(\mu, x) \geq \lambda$ for $x \in B$, then $\mu(B) \geq \lambda \mathcal{H}_\alpha(B)$.*
3. *If $\underline{D}^\alpha(\mu, x) \leq \lambda$ for $x \in B$, then $\mu(B) \leq \lambda \mathcal{P}^\alpha(B)$.*
4. *If $\underline{D}^\alpha(\mu, x) \geq \lambda$ for $x \in B$, then $\mu(B) \geq \lambda \mathcal{P}^\alpha(B)$.*

(See pages 95-97 in [24] for the proof.)

A set $E \subset \mathbb{R}^n$ is called m**-rectifiable** if there exist Lipschitz maps $f_i : \mathbb{R}^m \to \mathbb{R}^n$, $i = 1, 2..$, such that $\mathcal{H}_m(E \setminus \cup f_i(\mathbb{R}^m)) = 0$. A set $F \subset \mathbb{R}^n$ is called **purely** m**-rectifiable** if $\mathcal{H}_m(E \cap F) = 0$ for every m-rectifiable set E. A Radon measure μ on $\mathbb{R}^n$ is said to be m-rectifiable if $\mu \ll \mathcal{H}_m$, that is, μ is absolutely continuous with respect to the m-dimensional Hausdorff measure $\mathcal{H}_m$, and there exists an m-rectifiable Borel set E such that $\mu(\mathbb{R}^n \setminus E) = 0$.

Remark 1.1.13. *1. For $E \subset \mathbb{R}^n$, $\mathcal{L}_n(E) = c_n \mathcal{H}_n(E)$, where $\mathcal{L}_n$ denotes the n-dimensional Lebesgue measure and $c_n = \frac{\pi^{\frac{n}{2}} 2^{-n}}{(\frac{n}{2})!}$. [Refer [12] for the proof]*

2. *$\mathcal{H}_m$ is a constant multiple of the m-dimensional Lebesgue measure on sets which are m-rectifiable in $\mathbb{R}^n$ for all integers $1 \leq m < n$. [Refer [13] for the proof.]*

3. *Let $E \subset \mathbb{R}^n (0 < s < n)$ be a non-zero finite s-packing measurable set. Then $\mathcal{P}^s(E) = \mathcal{H}_s(E)$ if and only if s is an integer and $\mathcal{P}^s|_E$ is s-rectifiable.[Refer [24] for the proof.]*

The **Fourier dimension** of a set $A \subset \mathbb{R}^n$, $\dim_F(A)$ is the unique number in $[0, n]$ such that for any $0 < \beta < \dim_F(A)$ there exists a non-zero Radon measure μ with support of μ in A and $|\hat{\mu}(x)| \leq |x|^{-\beta/2}$ for $x \in \mathbb{R}^n$ and that for $\dim_F(A) < \beta < n$, no such measure exists.

Remark 1.1.14. *We have for any Borel set $A \subset \mathbb{R}^n$, $\dim_F(A) \leq \dim_H(A)$. The inequality is often strict. The sets with $\dim_F(K) = \dim_H(K)$ are called* ***Salem sets****.*

Example 1.1.15. *We recollect examples of Salem sets and sets with different Hausdorff dimension and Fourier dimension:*

1. *The ternary Cantor set C has Fourier dimension 0 but Hausdorff dimension $\ln 2/\ln 3$ (See [19] for the proof).*

2. *If $\psi : [0, \infty] \to \mathbb{R}^n$ denotes the n-dimensional Brownian motion, then for any compact set $F \subset [0, \infty]$, the image $\psi(F)$ is almost surely a Salem set. (See pages 136-137, 180 in [24] for proof.)*

Chapter 2

L^p-Integrability of the Fourier transform of fractal measures

In this chapter, we study the L^p-integrability of the Fourier transform of measures supported on sets of dimension $0 \leq \alpha < n$. We start by discussing the relation between α-regular sets and sets of finite α-packing measure. In the subsequent section, we prove that there does not exist any non zero function in $L^p(\mathbb{R}^n)$ with $1 \leq p \leq 2n/\alpha$ if its Fourier transform is supported by a set of finite packing α-measure where $0 < \alpha < n$. It is shown that this assertion fails for $p > 2n/\alpha$.

2.1 α-regular sets and sets of finite α-packing measure

The following lemma is crucial for us.

Lemma 2.1.1. *[37] Let $0 \leq \alpha < n$. Suppose $E \subset \mathbb{R}^n$ is such that $\mathcal{P}^\alpha(E) < \infty$ and $S \subset E$ is a bounded set. Then*

$$\limsup_{\epsilon \to 0} |S(\epsilon)| \epsilon^{\alpha - n} \leq C_n \mathcal{P}^\alpha(S) < \infty,$$

where $|S(\epsilon)|$ denotes the Lebesgue measure of ϵ-distance set of S, $S(\epsilon)$ and C_n is a constant which depends only on n.

Proof. Since $\mathcal{P}^\alpha(S) < \infty$, for a given $\delta > 0$, there exists a countable cover $\{\widetilde{A_i}\}$ of S such that

$\mathcal{P}^\alpha(S) + \delta = \sum P_0^\alpha(\widetilde{A_i}) < \infty$. Let $R > 0$ be such that $S \subset B_R(0)$. Then $\{A_i\}$ also covers S, where $A_i = \widetilde{A_i} \cap B_R(0)$ is bounded and $\sum P_0^\alpha(A_i) \leq \sum P_0^\alpha(\widetilde{A_i}) < \infty$. By Lemma 1.1.3,

$$\begin{aligned} |A_i(\epsilon)| &\leq \Omega_n(2\epsilon)^n N(A_i, \epsilon) \\ &\leq \Omega_n(2\epsilon)^n P(A_i, \epsilon/2) \\ &\leq \Omega_n 2^n \epsilon^{n-\alpha} P_\epsilon^\alpha(A_i). \end{aligned}$$

Hence $\epsilon^{\alpha-n}|A_i(\epsilon)| \leq C_n P_\epsilon^\alpha(A_i)$ for some fixed constant C_n. We also have $|S(\epsilon)| \leq \sum |A_i(\epsilon)|$. Hence, $\epsilon^{\alpha-n}|S(\epsilon)| \leq C_n \sum P_\epsilon^\alpha(A_i)$. So,

$$\limsup_{\epsilon\to 0} \epsilon^{\alpha-n}|S(\epsilon)| \leq C_n \sum P_0^\alpha(A_i) = C_n(\mathcal{P}^\alpha(S) + \delta) < \infty.$$

Hence letting δ to zero,

$$\limsup_{\epsilon\to 0} |S(\epsilon)|\epsilon^{\alpha-n} \leq C_n \mathcal{P}^\alpha(S) < \infty.$$

$\square$

Remark 2.1.2. *By Lemma 1.1.12, if μ is a Radon measure and E is quasi α-regular with respect to μ, that is, if there exists a non-zero constant λ such that $\lambda \leq \underline{D}^\alpha(\mu, x)$ for μ-almost all $x \in E$, then $\lambda\mathcal{P}^\alpha(A) \leq \mu(A)$ for all $A \subset E$.*

Lemma 2.1.3. *Let $0 \leq \alpha < n$ and μ be a Radon measure. If E is quasi α-regular with respect to μ, that is, if there exists a non-zero constant λ such that $\lambda \leq \underline{D}^\alpha(\mu, x)$ for μ-almost all $x \in E$, then for all bounded subsets S of E, we have*

$$\limsup_{\delta\to 0} |S(\delta)|\delta^{\alpha-n} \leq C_n \lambda^{-1}\mu(S),$$

where $|S(\delta)|$ denotes the n-dimensional Lebesgue measure of δ-distance set, $S(\delta)$ of S and C_n depends only on n.

Proof. The proof follows from the Remark 2.1.2 and Lemma 2.1.1. $\square$

We give an example of a set of finite α-packing measure and finite α-dimensional Hausdorff measure but not quasi regular. Before we explain the construction, let us recall the following:

Suppose $E \times F$ denotes the cartesian product of two non-empty Borel sets E and F in $\mathbb{R}^n$, then we have

$$\begin{aligned} \dim_H(E) + \dim_H(F) &\leq \dim_H(E \times F) \leq \dim_H(E) + \overline{\dim}_P(F) \\ &\leq \overline{\dim}_P(E \times F) \leq \overline{\dim}_P(E) + \overline{\dim}_P(F), \qquad (2.1.0.1) \\ \dim_H(E \times F) &= \dim_H(E) + \dim_H(F) \text{ if } \dim_H(F) = \overline{\dim}_P(F). \quad (2.1.0.2) \end{aligned}$$

(See page 115 in [24] page 72 in [43] for proof and examples that prove that the inequalities can be strict.)

Also if K is α-regular, then

$$\dim_H(K \times ... \times K) = n \dim_H(K). \qquad (2.1.0.3)$$

From Example 1.1.4, we can construct sets of finite α-packing measure but not quasi-regular with zero α-dimensional Hausdorff measure. The authors in [17] (Proposition 1 in Section 4) constructed a set $\widetilde{E}$ which is not β-quasi regular ($0 < \beta = \ln 2/\ln 3 < 1$). Similar to that we construct a set of finite packing measure and finite Hausdorff measure but not quasi regular.

Given a positive integer k, remove $2^k - 1$ intervals of equal length from $[0, 1]$ leaving 2^k subintervals of length 3^{-k}. Note that the length of each of the removed intervals is $\frac{3^k - 2^k}{3^k(2^k-1)}$. Repeat the excision on each of the 2^k subintervals leaving 2^{2k} subintervals of length 3^{-2k}. Then at the l^{th} stage we obtain a set K_l with 2^{kl} subintervals, each of length 3^{-kl}. Let $C(2^k, 3^k) = \cap_l K_l$. Note that $C(2, 3)$ is the Cantor set. For every k, this set $C(2^k, 3^k)$ has Hausdorff dimension $\beta = \ln 2/\ln 3$. In fact $C(2^k, 3^k)$ are β-regular sets with $\mathcal{H}_\beta(C(2^k, 3^k)) = 1$. By Theorem 1.1.7, $3^{-k\beta} \leq \underline{D}^\alpha(\mu, x)$ for all $x \in C(2^k 3^k)$. By Remark 2.1.2, $\mathcal{P}^\beta(C(2^k, 3^k)) \leq 3^{k\beta} < \infty$. Now, let

$$\widetilde{E}_j = \left[3^{-(j(j-1)/2)} C(2^j, 3^j) + 1 - 3^{-(j(j-1)/2)}\right] \backslash [1 - 3^{-(j(j+1)/2)}, 1],$$

where $3^{-(j(j-1)/2)} C(2^j, 3^j) + 1 - 3^{-(j(j-1)/2)}$ is obtained by dilating $C(2^j, 3^j)$ by $3^{-(j(j-1)/2)}$ and then translating by $1 - 3^{-(j(j-1)/2)}$. Note that $\widetilde{E}_j$'s are disjoint. Let $\widetilde{E}$ be the limit set $\cup_k \widetilde{E}_k$. Then for $\beta = \ln 2/\ln 3$, $0 < \mathcal{H}_\beta(\widetilde{E}) < \infty$. But $\underline{D}^\beta(\mu, x)$ goes to zero as x approaches 1 for $\mu = \mathcal{H}_\beta|_{\widetilde{E}}$.

However we note the following:

$$
\begin{aligned}
\mathcal{P}^{\beta}(\widetilde{E}) &\leq \sum_{j=1}^{\infty} 3^{j\beta}\mathcal{H}_{\beta}(\widetilde{E}_j) \\
&\leq \sum_{j=1}^{\infty} 3^{j\beta}3^{-\beta(j(j-1)/2)}(1-2^{-j}) \\
&\leq 3^{\beta}1/2 + 3^{\beta}\sum_{j=1}^{\infty} 3^{-\beta(j-1))} \\
&= 3^{\beta}(\frac{1}{2}+\frac{1}{3^{\beta}-1}) < \infty.
\end{aligned}
$$

It can be proved that the cartesian product $E = \widetilde{E} \times ... \times \widetilde{E}(n\ times)$ has non-zero finite α-dimensional Hausdorff measure, $\mathcal{P}^{\alpha}(E) < \infty$ but not quasi α-regular, for $\alpha = n\beta$.

In general, for given $0 < \alpha < n$, fix a large positive integer N and small number $\eta < 1$ such that $N\eta^{\beta} = 1$, where $n\beta = \alpha$. Then we can construct $C(N^k, \eta^{-k})$ as above and prove that for given α there exists a set E of Hausdorff and packing dimension α, such that E is of finite α-packing measure and α-dimensional Hausdorff measure but not quasi α-regular.

2.2 L^p-Integrability of the Fourier transform of fractal measures

In this section, we relate the fractal dimension of the support of the Fourier transform of a function on $\mathbb{R}^n$ with its membership in $L^p(\mathbb{R}^n)$ by proving that the Fourier transform of a tempered distribution supported in a fractal of dimension α ($0 \leq \alpha < n$) does not belong to $L^p(\mathbb{R}^n)$ for $1 \leq p \leq 2n/\alpha$. With an example of Salem set, we prove that the assertion fails for $p > 2n/\alpha$.

In [2], M. L. Agranovsky and E. K. Narayanan have related the integer dimension of the support of the Fourier transform of a function with its membership in L^p:

Theorem 2.2.1. *[2] If $f \in L^p(\mathbb{R}^n)$ and supp $\hat{f}$ is carried by a C^1-manifold M of dimension $d < n$ then $f = 0$ provided $1 \leq p \leq 2n/d$. If $d = 0$ then $f = 0$ for $1 \leq p < \infty$*

Also, the older result of Beurling in [6] gives an analogue statement of the above theorem for fractional dimensional sets in $\mathbb{R}$:

Theorem 2.2.2. *[6] If $f \in L^p(\mathbb{R})$, $p > 2$ and the Fourier transform of f is supported by a set of Hausdorff dimension $< 2/p$, then the function is identically zero.*

Let $A \subseteq \mathbb{R}^n$. A is called a **sparse set** or **thin set** if the n-dimensional Lebesgue measure of A, $|A|$ is zero. For $1 \leq p \leq 2$, if the Fourier transform of $f \in L^p(\mathbb{R}^n)$ is supported in a set of Lebesgue measure zero, then it is trivial that $f \equiv 0$. So we concentrate on the case $p > 2$. If $p > 2$, then $\widehat{f}$ is a tempered distribution and the support of $\widehat{f}$ is a closed set which may be thin. We closely follow the arguments in [1] (also see page 174 of [16]) and prove the following Lemma.

Lemma 2.2.3. *Let $f \in L^p(\mathbb{R}^n)$ with $2 \leq p \leq 2n/\alpha$, for some $0 < \alpha < n$. Let $\chi \in C_c^\infty(R^n)$ be supported in unit ball and $\int_{\mathbb{R}^n} \chi(x)dx = 1$. Denote $\chi_\epsilon(x) = \epsilon^{-n}\chi(x/\epsilon)$ and $u_\epsilon = u * \chi_\epsilon$ where $u = \widehat{f}$. Then*

$$\|u_\epsilon\|_2^2 \leq C\epsilon^{\alpha-n}\rho_\epsilon$$

where ρ_ϵ approaches 0 as ϵ tends to zero.

Proof. By the Plancherel theorem,

$$\begin{aligned}
\|u_\epsilon\|^2 &= \int_{\mathbb{R}^n} |f(x)|^2 |\widehat{\chi}(\epsilon x)|^2 \, dx = \sum_{j=-\infty}^{\infty} \int_{2^j \leqslant |\epsilon x| \leqslant 2^{j+1}} |f(x)|^2 |\widehat{\chi}(\epsilon x)|^2 \, dx \\
&\leqslant C\epsilon^{\alpha-n} \sum_{j=-\infty}^{\infty} 2^{j(n-\alpha)} \sup_{2^j \leqslant |\epsilon x| \leqslant 2^{j+1}} |\widehat{\chi}(\epsilon x)|^2 (2^{-j}\epsilon)^{n-\alpha} \int_{2^j \leqslant |\epsilon x| \leqslant 2^{j+1}} |f(x)|^2 dx \\
&= C\epsilon^{\alpha-n} \sum_{j=-\infty}^{\infty} a_j b_j^\epsilon,
\end{aligned}$$

where

$$a_j = 2^{j(n-\alpha)} \sup_{2^j \leqslant |x| \leqslant 2^{j+1}} |\widehat{\chi}(x)|^2,$$

and

$$b_j^\epsilon = (2^{-j}\epsilon)^{n-\alpha} \int_{2^j \leqslant |\epsilon x| \leqslant 2^{j+1}} |f(x)|^2 dx.$$

Since $0 < \epsilon < 1$ and $2 \leq p \leq 2n/\alpha$, applying Holder's inequality,

$$|b_j^\epsilon| \leqslant C \left(\int_{2^j \epsilon^{-1} \leqslant |x| \leqslant 2^{j+1}\epsilon^{-1}} |f(x)|^p dx \right)^{2/p},$$

which goes to zero as $\epsilon \to 0$, for any fixed j. Also we have $|b_j^\epsilon| \leqslant C\|f\|_p^2 < \infty$ for some constant C independent of ϵ and j. Since $\sum_j |a_j|$ is finite, by the dominated convergence theorem, we have $\rho_\epsilon = \sum_j a_j b_j^\epsilon \to 0$ as $\epsilon \to 0$.

□

Theorem 2.2.4. *[37] Let $f \in L^p(\mathbb{R}^n)$ be such that supp $\widehat{f}$ is contained in a set E of finite α-dimensional packing measure. Then $f \equiv 0$, provided $p \le \frac{2n}{\alpha}$.*

Proof. By convolving f with a compactly supported smooth function we can assume that $f \in L^p(\mathbb{R}^n)$ where $p = 2n/\alpha$. Choose an even function $\chi \in C_c^\infty(R^n)$ with support in unit ball and $\int_{\mathbb{R}^n} \chi(x)dx = 1$. Let $\chi_\epsilon(x) = \epsilon^{-n}\chi(x/\epsilon)$ and $u_\epsilon = u * \chi_\epsilon$ where $u = \widehat{f}$. Then by Lemma 2.2.3,

$$\|u_\epsilon\|_2^2 \le C\epsilon^{\alpha-n}\rho_\epsilon$$

where ρ_ϵ approaches 0 as ϵ tends to zero. Let $\psi \in C_c^\infty(\mathbb{R}^n)$. Let $S = supp\ \widehat{f} \cap supp\ \psi$. Then S is a closed and bounded subset of E and hence $\mu(S) < \infty$ (μ is a Radon measure). By the Lemma 2.1.1,

$$\limsup_{\epsilon\to 0} \epsilon^{\alpha-n}|S_\epsilon| < \mu(S) < \infty.$$

So,

$$\begin{aligned}
| < u, \psi > |^2 &= \lim_{\epsilon\to 0} | < u_\epsilon, \psi > |^2 \\
&\le \lim_{\epsilon\to 0} \|u_\epsilon\|_2^2 \int_{S_\epsilon} |\psi|^2 \\
&\le \lim_{\epsilon\to 0} C\epsilon^{\alpha-n} \sum_{j=-\infty}^{\infty} a_j b_j^\epsilon \int_{S_\epsilon} |\psi|^2 \\
&\le C'\|\psi\|_\infty^2 \lim_{\epsilon\to 0} \epsilon^{\alpha-n}|S_\epsilon|\rho_\epsilon \\
&= 0
\end{aligned}$$

Hence $f = 0$.

□

From Lemma 2.1.3 and Theorem 2.2.4, we have,

Corollary 2.2.5. *Let $f \in L^p(\mathbb{R}^n)$ be such that supp $\widehat{f}$ is contained in a quasi α-regular set E that has non-zero finite α-dimensional Hausdorff measure. Then $f \equiv 0$, provided $p \le \frac{2n}{\alpha}$.*

Any d-dimensional C^1-smooth manifold M is d-rectifiable and $\mathcal{P}^d|_M$ is d-rectifiable. By Remark 1.1.13, $\mathcal{P}^d(M) = \mathcal{H}_d(M)$ and thus $\mathcal{P}^d$ is a constant multiple of d-dimensional Lebesgue measure. Since Lebesgue measure is locally finite, we can assume M to have finite d-packing measure. Also by Lemma 1.1.10, the d-density $D^d(\mu, x) = \underline{D^d}(\mu, x) = \overline{D^d}(\mu, x) = 1$, where μ denotes the d-dimensional Lebesgue measure restricted to M. Hence M is quasi d-regular. Hence Theorem 2.2.4 extends Theorem 2.2.1.

Remark 2.2.6. *For a set E such that $\mathcal{P}^\alpha(E) < \infty$, we have $\overline{\dim}_P(E) \leq \alpha$. Also, if $\overline{\dim}_P(E) < \overline{\dim}_M(E) = \alpha$, then $\mathcal{P}^\alpha(E) < \infty$. Thus if E has upper Minkowski dimension α and $f \in L^p(\mathbb{R}^n)$ with support of its Fourier transform supported in E ,then $f \equiv 0$ provided $p < 2n/\alpha$.*

Corollary 2.2.7. *Let $f \in L^p(\mathbb{R}^n)$ be such that supp $\widehat{f}$ is contained in a set E where $\overline{\dim}_P(E) = \alpha$. Then $f \equiv 0$, provided $p < \frac{2n}{\alpha}$.*

Proof. From (1.1.0.1), $\mathcal{P}^\beta(E) = 0$ for all $\beta > \alpha$. Hence by the Corollary 2.2.5, $f \equiv 0$, provided $p \leq \frac{2n}{\beta}$ for all $\beta > \alpha$. Thus $f \equiv 0$, provided $p < \frac{2n}{\alpha}$. $\square$

If $\dim_H(E) \leq \alpha = \overline{\dim}_P(E)$, then Beurling's Theorem 2.2.2 implies Corollary 2.2.7. However, with a weaker hypothesis, Corollary 2.2.5 strengthens Beurling Theorem 2.2.2.

Next we show that Corollary 2.2.5 is sharp and hence the sharpness of the Theorem 2.2.4. First, let us recall a well known example due to Salem which shows that there exists a measure ν supported on a Cantor type set $K \subseteq \mathbb{R}$, of Hausdorff dimension $\beta,\ 0 < \beta < 1$ with Fourier tranform $\widehat{\nu}$ belonging to $L^q(\mathbb{R})$ for all $q > 2/\beta$ (See page 263-271 in [8]). Let $M = K \times K \times ... \times K$ (n times) and $\mu = \nu \times \nu \times ... \times \nu$ (n times). Then μ is supported in M and $\widehat{\mu} \in L^q(\mathbb{R}^n)$ for $q > \frac{2}{\beta} = \frac{2n}{\alpha}$ where $\alpha = n\beta$. Closely following the proof in page 33 in [8] we show that not only the Hausdorff dimension of M is α, but M is also Ahlfors-David regular (hence quasi α-regular) set of finite Hausdorff measure, $\mathcal{H}_\alpha$. Then by Lemma 1.1.12, M is of finite α-packing measure. Thus the range in Theorem 2.2.4 is the best possible.

First, we briefly recall how the above set $K \subseteq \mathbb{R}$ is constructed. Choose a positive number η and an integer N so that $N\eta < 1$ and

$$N\eta^\beta = 1. \tag{2.2.0.1}$$

Choose N independent points a_i in the unit interval $[0,1]$ in such a way that $0 \leq a_1 < a_2 < ... < a_N \leq 1-\eta$ and widely enough spaced so that the distance between two a_i is larger than η. The set K is constructed as the intersection of decreasing sequence of compact sets $\mathcal{K}_j$, where $\mathcal{K}_j$ are defined as follows:

Choose an increasing sequence of non-zero positive numbers η_j converging to η where

$$\eta(1-\frac{1}{(j+1)^2}) \leq \eta_j \leq \eta \tag{2.2.0.2}$$

for all j. The first set, K_1, is the union of N intervals of length η_1 of the form $[a_k, a_k+\eta_1]$. The second set K_2, has N^2 intervals of length $\eta_1\eta_2$ of the form $[a_i + a_j\eta_1, a_i + a_j\eta_1 + \eta_1\eta_2]$ and so on. Inductively, we obtain a sequence K_j of decreasing sets of length $\eta_1\eta_2...\eta_j$. Then $K = \cap_j K_j$. It is known that the Hausdorff dimension of K is β. (see [34] and page 268 in [8])

Lemma 2.2.8. *[37] Hausdorff dimension of $M = K \times K \times .. \times K$ (n times) equals $\alpha = n\beta$ and M is an Ahlfors-David α-regular set.*

Proof. Let $0 < r < 1$ and $x \in M$, that is let $x = (x_1, x_2, ..., x_n)$ where $x_m \in K$ for all m. For every m, by construction of K, there exists a smallest integer t_m such that $K \cap (x_m - r, x_m + r)$ contains at least one interval I_{t_m} of length $\eta_1..\eta_m$. Thus

$$K \cap (x_m - r, x_m + r) \supseteq K \cap I_{t_1} \times ... \times K \cap I_{t_n}. \tag{2.2.0.3}$$

Since Hausdorff measure is translation invariant, we can assume $2r \leq \eta_1...\eta_{t_m-1}$. Since $\alpha = n\beta$,

$$(2r)^\alpha \leq \Pi_{m=1}^n(\eta_1^\beta...\eta_{t_m-1}^\beta). \tag{2.2.0.4}$$

Among the coverings of $M \cap B_r(x)$ which compete in the definition of $\mathcal{H}_\alpha(M \cap B_r(x))$, are the coverings M_j(where $j = (j_1, ..., j_n)$ and large $j_m > t_m - 1$) themselves, consisting of $\Pi_m^n N^{j_m-(t_m-1)}$ cubes of volume $\Pi_{m=1}^n(\eta_1\eta_2...\eta_{j_m})$. Hence

$$\begin{aligned}
\mathcal{H}_\alpha(M \cap B_r(x)) &\leq \Pi_{m=1}^n N^{j_m-(t_m-1)}(\eta_1\eta_2...\eta_{j_m})^\beta \\
&\leq (2r)^\alpha \Pi_{m=1}^n N^{j_m-(t_m-1)}(\eta_{t_m+1}\eta_2...\eta_{j_m})^\beta \text{ from (2.2.0.3)} \\
&\leq (2r)^\alpha N^{-n}\Pi_{m=1}^n N^{j_m-t_m}\eta^{(j_m-t_m)\beta} \text{ from (2.2.0.2)}
\end{aligned}$$

Thus from (2.2.0.1) we have

$$\mathcal{H}_\alpha(M \cap B_r(x)) \leq \frac{2^\alpha}{N^n} r^\alpha \tag{2.2.0.5}$$

Similarly we prove that $\mathcal{H}_\alpha(M) \leq 1$ which implies that the Hausdorff dimension of M is at most α. To show that the dimension of M is exactly α, we show that $\mathcal{H}_\alpha(M)$ is not 0.

In computing the Hausdorff measure, it is enough to take the infimum of Σd_i^α over all coverings of $M \cap B_r(x)$ by countable families of (sufficiently small) open balls A_i, where the end points of the projection of A_i to m^{th} axis is in the complement of $K \cap (x_m - r, x_m + r)$. From the compactness, it is also clear that these coverings consist of only a finite number of disjoint, open cubes. Let $\{U_i\}$ be one such family of sufficiently small cubes that cover $M \cap B_r(x)$, where the end points of the projection of U_i to m^{th} axis is in the complement of $K \cap (x_m - r, x_m + r)$.

Let p_{i_m} be the smallest integer p such that m^{th} projection of U_i contains at least one interval of K_p and $P_i = (p_{i_1}, p_{i_2}, ...p_{i_n})$. Then, from (2.2.0.3), $t_m \leq p_{i_m}$. Let

$$p_{i_m} = t_m + s_{i_m} \tag{2.2.0.6}$$

Let m^{th} projection of U_i contain $k_i^{(m)}$ number of constituent intervals of K_{p_m}. Then U_i contain $k_i = \Pi_{m=1}^n k_i^{(m)}$ number of cubes of $M_{P_i} = K_{p_{i_1}} \times ... \times K_{p_{i_n}}$. Let d_i denote the diameter of U_i. Then

$$d_i^n \geq k_i \Pi_{m=1}^n (\eta_1 \eta_2 ... \eta_{p_{i_m}}). \tag{2.2.0.7}$$

Let j_m's be large such that $\cup U_i$ contains $M_j \cap M \cap B_r(x)$ where $M_j = K_{j_1} \times ... \times K_{j_n}$ and $M_j \subset M_{P_i}$, for all i. Then U_i contains $k_i N^{(j_1 - p_{i_1} + ... + j_n - p_{i_n})}$ cubes of M_j. By (2.2.0.6), U_i contains $k_i N^{(j_1 - t_1 - s_{i_1} + ... + j_n - t_n - s_{i_n})}$ cubes of M_j. However by (2.2.0.3),

$$M \cap B_r(x) \cap M_j \subseteq M_t \subset M \cap B_r(x),$$

where $M_t = (K \cap I_{t_1}) \times ... \times (K \cap I_{t_n})$. So the number of cubes of M_j covered by $\cup U_i$ is at least $N^{j_1 - t_1 + ... + j_n - t_n}$. Since $\sum_i k_i N^{(j_1 - t_1 - s_{i_1} + ... + j_n - t_n - s_{i_n})}$ is the total number of cubes of M_j covered by $\cup U_i$,

$$\sum_i k_i N^{(j_1 - t_1 - s_{i_1} + ... + j_n - t_n - s_{i_n})} \geq N^{j_1 - t_1 + ... + j_n - t_n}. \tag{2.2.0.8}$$

The equation (2.2.0.7) implies that

$$\begin{aligned} d_i^\alpha &\geq (k_i \Pi_{m=1}^n (\eta_1 \eta_2 ... \eta_{p_{i_m}}))^\beta \\ &\geq (2r)^\alpha (k_i \Pi_{m=1}^n (\eta_{t_m} \eta_{t_m+1} ... \eta_{p_{i_m}}))^\beta (\text{from } (2.2.0.4)) \\ &\geq (2r)^\alpha (k_i \Pi_{m=1}^n \eta_{t_m} \eta^{p_{i_m}-t_m} [(1 - \frac{1}{(t_m+1)^2})...(1-\frac{1}{p_{i_m}^2})])^\beta \text{ from } (2.2.0.2) \end{aligned}$$

Since η_m is an increasing sequence and by (2.2.0.2), $\eta_{t_1}\eta_{t_2}...\eta_{t_n} \geq (\frac{3}{4}\eta)^n$. Fix $C = (\frac{3}{4}\eta)^n$. Thus

$$\begin{aligned} d_i^\alpha &\geq C(2r)^\alpha \big(k_i \eta^{(p_{i_1}+...+p_{i_n}-(t_1+...t_n))} \Pi_{m=1}^n \big[(1-\frac{1}{t_m+1})(1+\frac{1}{p_{i_m}})\big]\big)^\beta \\ &\geq C(2r)^\alpha \big(k_i \eta^{(p_{i_1}+...+p_{i_n}-(t_1+...t_n))} \Pi_{m=1}^n \big[\frac{1}{2}(1+\frac{1}{p_{i_m}})\big]\big)^\beta \\ &> C r^\alpha k_i^\beta \eta^{(p_{i_1}+...+p_{i_n}-(t_1+...t_n))\beta}, \end{aligned}$$

From (2.2.0.1), we have

$$N^{(j_1+...j_n)-(p_{i_1}+...p_{i_n})} \eta^{(j_1+...j_n-t_1-...t_n)\beta} = \eta^{(p_{i_1}+...+p_{i_n}-(t_1+...t_n))}.$$

Thus

$$d_i^\alpha \geq C r^\alpha k_i^\beta N^{(j_1+...j_n)-(p_{i_1}+...p_{i_n})} \eta^{(j_1+...j_n-t_1-...t_n)\beta}. \tag{2.2.0.9}$$

Also, there exists a constant $C_{N,n}$ $(= 2^n(N-1)^n)$, such that $1 \leq k_i \leq C_{N,n}$ because of the choice of p_{i_k}. Let $L = (C_{N,n})^{\beta-1}$. Since $0 < \beta < 1$,

$$k_i^\beta > L k_i \tag{2.2.0.10}$$

From (2.2.0.6) and (2.2.0.10), summing over i in (2.2.0.9), we have

$$\begin{aligned} \Sigma_i d_i^\alpha &\geq C L r^\alpha \eta^{(j_1+...j_n-t_1-...-t_n)\beta} \Sigma_i k_i N^{(j_1+...j_n)-(t_1+...t_n+s_{i_1}+...s_{i_n})} \\ &\geq C L r^\alpha \eta^{(j_1+...j_n-t_1-...-t_n)\beta} N^{j_1+...j_n-t_1-...-t_n} \text{ (from (2.2.0.8))} \\ &= C L r^\alpha \text{ (from (2.2.0.1))} \\ &> 0 \end{aligned}$$

Thus

$$\mathcal{H}_\alpha(M \cap B_r(x)) \geq C L r^\alpha \tag{2.2.0.11}$$

for all $x \in M$ and $0 < r < 1$. Similarly we prove that $\mathcal{H}_\alpha(M) > 0$. From (2.2.0.5) and (2.2.0.11) we have proved that there exists non-zero finite constants a and b such that

$$0 < a r^\alpha \leq \mathcal{H}_\alpha(M \cap B_r(x)) \leq b r^\alpha < \infty.$$

for all $x \in M$ and $0 < r < 1$. Hence the proof. □

Remark 2.2.9. *The set constructed in the above Lemma 2.2.8 is a fractal even if* α *is an integer. In [2], the authors proved the sharpness of Theorem 2.2.1 for any integer* $\alpha \geq n/2$ *by constructing a smooth manifold* $M \subset \mathbb{R}^n$ *and* μ *supported on* M *such that the Fourier transform* $f = \hat{\mu} \in L^p(\mathbb{R}^n)$ *for all* $p > 2n/\alpha$*. The case* $0 < \alpha < n/2$*,* α *integer seems to be still open.*

Chapter 3

L^p-Asymptotics of Fractal measures

In this chapter, we give quantitative versions of the Theorem 2.2.4 proved in the previous chapter by estimating the L^p norm of the Fourier transform of fractal measures μ over a ball centered at origin with large radius for $1 \leq p \leq 2n/\alpha$ under various fractal geometric assumption on the support of μ. We study the L^p-asymptotics of the Fourier transform of the fractal measures for $2 \leq p \leq 2n/\alpha$ in the next section and for $1 \leq p \leq 2$ in the subsequent section.

3.1 L^p-Fourier asymptotic properties of fractal measures for $2 \leq p \leq 2n/\alpha$

Let μ denote a fractal measure supported in an α-dimensional set $E \subset \mathbb{R}^n$ and $f \in L^q(d\mu)$ $(1 \leq q \leq \infty)$. Suppose $2 < p \leq 2n/\alpha$. In this section, we obtain the upper and lower bounds for

$$\frac{1}{L^{n-\frac{\alpha p}{2}}} \int_{|\xi| \leq L} |\widehat{f d\mu}(\xi)|^p d\xi. \tag{3.1.0.1}$$

Strichartz proved in [41] an analogue of Radon-Nikodym theorem for positive measure with no infinite atoms:

Theorem 3.1.1. *[41] Let μ be a measure with no infinite atoms, and let ν be σ-finite and absolutely continuous with respect to μ. Then there exists a unique decomposition $\nu = \nu_1 + \nu_2$ such that*

$d\nu_1 = \phi d\mu$ for a non-negative measurable function ϕ and ν_2 is null with respect to μ, that is, $\nu_2(A) = 0$ whenever $\mu(A) < \infty$.

Remark 3.1.2. *As observed in [41], any locally uniformly α-dimensional measure μ can be written as $d\mu = \phi d\mathcal{H}_\alpha + d\nu$ where ν is null with respect to $\mathcal{H}_\alpha$ and ϕ is a non negative measurable function belonging to $L^1(\mathbb{R}^n)$.*

In [41], Strichartz studied the asymptotic properties of locally uniformly α-dimensional measures and proved a Plancherel type theorem:

Theorem 3.1.3. *[41]*

1. *Let $d\mu = \phi d\mathcal{H}_\alpha + d\nu$ be a locally uniformly α-dimensional measure on $\mathbb{R}^n$ (as in Remark 3.1.2). For any $f \in L^2(d\mu)$ we have, for a fixed y and a constant c independent of y,*
$$\limsup_{L\to\infty} \frac{1}{L^{n-\alpha}} \int_{B_L(y)} |\widehat{(f d\mu)}|^2 \leq c \int |f(x)|^2 \phi(x) d\mathcal{H}_\alpha(x).$$

2. *Let $\mu' = \mu + \nu$ be a locally uniformly α-dimensional measure on $\mathbb{R}^n$ where $\mu = \mathcal{H}_\alpha|_E$ and ν is null with respect to $\mathcal{H}_\alpha$. If E is quasi regular, then for fixed y and constant c independent of y,*
$$c \int_E |f|^2 d\mathcal{H}_\alpha \leq \liminf_{L\to\infty} \frac{1}{L^{n-\alpha}} \int_{B_L(y)} |\widehat{f d\mu'}|^2.$$

These results are analogous to the results proved by Agmon and Hormander in [1] when α is an integer.

Also, with the use of mean quadratic variation, Lau in [21] investigated the fractal measures by defining a class of complex valued σ-finite Borel measures μ on $\mathbb{R}^n$, $\mathcal{M}^p_\alpha$, for $1 \leq p < \infty$ with
$$\|\mu\|_{\mathcal{M}^p_\alpha} = \sup_{0<\delta\leq 1} \left(\frac{1}{\delta^{n+\alpha(p-1)}} \int_{\mathbb{R}^n} |\mu(Q_\delta(x))|^p \right)^{1/p} < \infty$$
and
$$\|\mu\|_{\mathcal{M}^\infty_\alpha} = \underset{u\in\mathbb{R}^n}{sup}\ \underset{0<\delta<1}{sup} \frac{1}{(2\delta)^\alpha} |\mu|(Q_\delta(u)) < \infty,$$
where $Q_\delta(u)$ denotes the half open cube $\prod_{j=1}^n (x_j - \delta, x_j + \delta]$. For $1 \leq p < \infty,\ 0 \leq \alpha < n$, $\mathcal{B}^p_\alpha$ denotes the set of all locally p-th integrable function f in $\mathbb{R}^n$ such that
$$\|f\|_{\mathcal{B}^p_\alpha} = \underset{L\geq 1}{sup} \Big(\frac{1}{L^{n-\alpha}} \int_{B_L} |f|^p \Big)^{1/p} < \infty.$$

For $0 \le \alpha \le \beta < n$, we have from [22], $\mathcal{B}^p_\beta \subseteq \mathcal{B}^p_\alpha \subseteq \mathcal{B}^p_0 \subseteq L^p(dx/(1+|x|^{n+1}))$. For $\delta > 0$, we define the transformation W_δ as

$$(W_\delta f)(x) = \int_{\mathbb{R}^n} f(y) E_\delta(y) e^{2\pi i x.y} dy,$$

where $E_\delta(y) = \int_{|\xi|\le\delta} e^{2\pi i y\xi} d\xi = 2\pi(\delta|y|^{-1})^{n/2} J_{n/2}(2\pi\delta|y|)$ and $J_{n/2}$ is the Bessel function of order $n/2$. If μ is a bounded Borel measure on $\mathbb{R}^n$ and $f = \widehat{\mu}$, then for $\delta > 0$ and for any ball $B_\delta(x)$, $\mu(B_\delta(x)) = (W_\delta f)(x)$ for Lebesgue almost all $x \in \mathbb{R}^n$. Lau studied the asymptotic properties of measures in $\mathcal{M}^p_\alpha$ in [21]:

Theorem 3.1.4. *[21]*

1. *Let $1 \le p \le 2$, $1/p + 1/p' = 1$ and $0 \le \alpha < n$. Suppose $\mu \in \mathcal{M}^p_\alpha$ then $\hat{\mu} \in \mathcal{B}^{p'}_\alpha$ with*

$$\sup_{L\ge1}\Big(\frac{1}{L^{n-\alpha}}\int_{B_L}|\hat{\mu}|^{p'}\Big)^{1/p'} = \|\hat{\mu}\|_{\mathcal{B}^{p'}_\alpha} \le C\|\mu\|_{\mathcal{M}^p_\alpha},$$

   for some constant C depending on μ.

2. *Let μ be a positive σ-finite Borel measure on $\mathbb{R}^n$ and f be any Borel μ-measurable function on $\mathbb{R}^n$. Let $d\mu_f = f d\mu$. μ is locally uniformly α-dimensional if and only if $\|\mu_f\|_{\mathcal{M}^p_\alpha} \le C\|f\|_{L^p(d\mu)}$ for all $f \in L^p(d\mu)$, $p > 1$ and C is a non-zero constant dependent on p.*

Applying Holder's inequality to part (2) of the Theorem 3.1.3, we obtain:

Corollary 3.1.5. *Let $f \in L^2(\mu)$ be supported in a quasi α-regular set E of non-zero finite α-dimensional Hausdorff measure ($0 < \alpha < n$), where μ is a locally uniformly α-dimensional measure. Then for $p \ge 2$,*

$$\|f\|^p_{L^2(\mu)} \le c \limsup_{L\to\infty} \frac{1}{L^{n-\frac{\alpha p}{2}}} \int_{|\xi|\le L} |\widehat{f d\mu}(\xi)|^p d\xi$$

where c is a non zero finite constant depending on n, α and p.

The above results are proved for locally uniformly α-dimensional measure. But if a set E is of finite α-packing measure, then $\mu = \mathcal{P}^\alpha|_E$ need not be locally uniformly α-dimensional measure. We prove an analogue result to the above corollary for the range $2 \le p < 2n/\alpha$ with $\mu = \mathcal{P}^\alpha|_E$, where $\mathcal{P}^\alpha(E) < \infty$.

Theorem 3.1.6. *Let $f \in L^2(d\mu)$ be a positive function where $\mu = \mathcal{P}^\alpha|_E$ and E is a compact set of finite α-packing measure. Then for $2 \le p < 2n/\alpha$,*

$$\int_{\mathbb{R}^n} |f(x)|^2 d\mu(x) \le \; C \liminf_{L\to\infty} \Big(\frac{1}{L^{n-\alpha p/2}} \int_{B_L(0)} |\widehat{fd\mu}(\xi)|^p d\xi\Big)^{2/p}$$

and

$$\int_{\mathbb{R}^n} |f(x)|^2 d\mu(x) \le \; C' \liminf_{L\to\infty} \Big(\frac{1}{L^{n-\alpha p/2}} \int_{\mathbb{R}^n} e^{-\frac{|\xi|^2}{2L^2}} |\widehat{fd\mu}(\xi)|^p d\xi\Big)^{2/p},$$

where the constants C and C' are independent of f.

Proof. Since E is compact, without loss of generality we assume that E is contained in a large cube in the positive quadrant, that is, there exists smallest positive integer m such that for all $x = (x_1, ...x_n) \in E$, $0 < x_j < m$. Let $M = \{x = (x_1, ...x_n) \in \mathbb{R}^n : 0 \le x_j \le m, \ \forall j\}$.

Fix $0 < \epsilon < 1$. For $k = (k_1, ...k_n)$, $(0 < k_j \in \mathbb{Z})$,

$$Q_k = \{x = (x_1, ...x_n) \in M : \ (k_j - 1)\epsilon < x_j \le k_j\epsilon, \ j = 1, ...n\}.$$

Let $\mathcal{Q}_0$ be the collection of all such Q_k's whose intersection with E has non-zero μ-measure, that is, $\mu(Q_k) \ne 0$. Since E is compact, there exists finite number of Q_k's in $\mathcal{Q}_0$. Let $\tilde{\delta_0} = \min_{Q_k\in\mathcal{Q}_0}\{\mu(Q_k)\}$. Then $E = \cup(Q_k \cap E) \cup E'$ where the union is finite and $\mu(E') = 0$.

$$\begin{aligned}
\int_E |f(x)|^2 d\mu(x) &= \sum_{Q_k\in\mathcal{Q}_0} \int_{Q_k} |f(x)|^2 d\mu(x) \\
&\le 2 \sum_{Q_k\in\mathcal{Q}_0} \int_{Q_k} \Big| f(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} f(y) d\mu(y)\Big|^2 d\mu(x) \\
&\quad + 2 \sum_{Q_k\in\mathcal{Q}_0} \frac{1}{\mu(Q_k)} \Big| \int_{Q_k} f(y) d\mu(y)\Big|^2. \qquad (3.1.0.2)
\end{aligned}$$

Now by Lemma 2.1.1, for each k, there exists δ_k such that

$$\begin{aligned}
|(Q_k \cap E)(\delta)|\delta^{\alpha-n} &\le C_n\mathcal{P}^\alpha(Q_k \cap E) + C_n\tilde{\delta_0}\epsilon \\
&\le 2C_n\mathcal{P}^\alpha(Q_k \cap E) = 2C_n\mu(Q_k), \qquad (3.1.0.3)
\end{aligned}$$

for all $\delta \le \delta_k$. Fix $\delta_0 = \min\{\epsilon, \ \tilde{\delta_0}, \ \delta_1, \ \delta_2, ..\}$. Since there are finite Q_k's, $\delta_0 > 0$. Let ϕ be a positive Schwartz function such that $\widehat{\phi}(0) = 1$, support of $\widehat{\phi}$ is supported in the unit ball and there

exists $r_1 > 0$ such that

$$\int_{A_{r_1}(0)} \phi(x)dx = \frac{1}{2^{n+1}}, \tag{3.1.0.4}$$

where $A_{r_1}(0) = \{x = (x_1, ...x_n) : -r_1 < x_j \leq 0, \; \forall \, j\}$. Denote $\phi_L(x) = \phi(Lx)$ for all $L > 0$. Let $r = n^{\frac{1}{2}} r_1$. Fix L large such that $r/L \leq \delta_0$. Then we have,

$$\begin{aligned}
\left| \int_{Q_k} f(y)d\mu(y) \right|^2 &= 2^{2(n+1)} \left| \int_{Q_k} \int_{A_{r_1}(0)} \phi(x)dx f(y)d\mu(y) \right|^2 \\
&= 2^{2(n+1)} L^{2n} \left| \int_{Q_k} \int_{A_{r_1/L}(y)} \phi_L(x-y)dx f(y)d\mu(y) \right|^2 \\
&= 2^{2(n+1)} L^{2n} \left| \int_{Q_k^L} \int_{Q_k} \phi_L(x-y)f(y)d\mu(y)dx \right|^2, \qquad (3.1.0.5)
\end{aligned}$$

where

$$Q_k^L = \{x = (x_1, ..x_n) \in \mathbb{R}^n : \exists \, y = (y_1, ...y_n) \in E \text{ such that } y_j - r_1/L < x_j \leq y_j, \; \forall \, j\}.$$

Then $|Q_k^L| \leq |(Q_k \cap E)(r/L)|$, where $(Q_k \cap E)(r/L)$ denotes the r/L-distance set of $Q_k \cap E$ (since $r = \sqrt{n} r_1$). Also since ϕ and f are positive,

$$\int_{Q_k^L} \int_{Q_k} \phi_L(x-y)f(y)d\mu(y)dx \leq \int_{Q_k^L} \phi_L * f d\mu(x)dx.$$

Thus from (3.1.0.5),

$$\begin{aligned}
\frac{1}{2^{2(n+1)}} \left| \int_{Q_k} f(y)d\mu(y) \right|^2 &\leq L^{2n} \left| \int_{Q_k^L} \phi_L * f d\mu(x)dx \right|^2. \\
&\leq L^{2n} |Q_k^L| \int_{Q_k^L} |\phi_L * f d\mu(x)|^2 dx \\
&\leq L^{2n} |(Q_k \cap E)(r/L)| \int_{Q_k^L} |\phi_L * f d\mu(x)|^2 dx \\
&\leq 2C_n r^{n-\alpha} L^{n+\alpha} \mu(Q_k) \int_{Q_k^L} |\phi_L * f d\mu(x)|^2 dx \text{ by (3.1.0.3).}
\end{aligned}$$

Thus there exists a constant C_1 independent of ϵ, L and f such that

$$\frac{1}{\mu(Q_k)} \left| \int_{Q_k} f(y)d\mu(y) \right|^2 \leq C_1 L^{n+\alpha} \int_{Q_k^L} |\phi_L * f d\mu(x)|^2 dx.$$

Hence, from (3.1.0.2)

$$\int_E |f(x)|^2 d\mu(x) \leq 2\sum_k \int_{Q_k} \left| f(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} f(y)d\mu(y) \right|^2 d\mu(x) + 2C_1 L^{n+\alpha} \sum_{Q_k \in \mathcal{Q}} \int_{Q_k^L} |\phi_L * f d\mu(x)|^2 dx.$$

By the choice of $r/L < \delta_0 < \epsilon$, any $x \in Q_k^L$ is intersected at most 2^n number of other Q_k^L's in $\mathcal{Q}_0$. Hence there exists a constant $C = 2C_1 2^n$ independent of f, ϵ and L such that for all $r/L \leq \delta_0$

$$\int_E |f(x)|^2 d\mu(x) \leq 2e_\epsilon + CL^{n+\alpha} \int_{E(r/L)} |\phi_L * f d\mu(x)|^2 dx, \tag{3.1.0.6}$$

where

$$e_\epsilon = \sum_k \int_{Q_k} |f(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} f(y)d\mu(y)|^2 d\mu(x).$$

For given ϵ, let $g \in C_c^\infty(d\mu)$ be such that $\|f-g\|^2_{L^2(d\mu)} < \epsilon$. Then,

$$\begin{aligned}
e_\epsilon &= \sum_k \int_{Q_k} \left| f(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} f(y)d\mu(y) \right|^2 d\mu(x) \\
&\leq 2\sum_k \int_{Q_k} \left| f(x) - g(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} f(y) - g(y)d\mu(y) \right|^2 d\mu(x) \\
&\quad +2\sum_k \int_{Q_k} \left| g(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} g(y)d\mu(y) \right|^2 d\mu(x) \\
&\leq 4\sum_k \int_{Q_k} |f(x) - g(x)|^2 + \left| \frac{1}{\mu(Q_k)} \int_{Q_k} f(y) - g(y)d\mu(y) \right|^2 d\mu(x) \\
&\quad +2\sum_k \int_{Q_k} \left| g(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} g(y)d\mu(y) \right|^2 d\mu(x) \\
&\leq 8\sum_k \int_{Q_k} |f(x) - g(x)|^2 d\mu(x) \\
&\quad +2\sum_k \int_{Q_k} \left| g(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} g(y)d\mu(y) \right|^2 d\mu(x).
\end{aligned}$$

Since $E = \cup_k (Q_k \cap E) \cup E'$ and $\mu = \mathcal{P}^\alpha|_E$,

$$\begin{aligned}
e_\epsilon &\leq 8\|f-g\|^2_{L^2(d\mu)} + 2\sum_k \int_{Q_k} \left| g(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} g(y)d\mu(y) \right|^2 d\mu(x) \\
&\leq \epsilon + 2\sum_k \int_{Q_k} \left| g(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} g(y)d\mu(y) \right|^2 d\mu(x).
\end{aligned} \tag{3.1.0.7}$$

Since g is compactly supported continuous function, g is uniformly continuous and

$$|g(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} g(y) d\mu(y)| \to 0$$

uniformly in x and Q_k as $\mu(Q_k) \to 0$. As $\epsilon \to 0$, we have $\mu(Q_k) \to 0$. Hence

$$\begin{aligned} & \sum_k \int_{Q_k} \left| g(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} g(y) d\mu(y) \right|^2 d\mu(x) \\ \leq \ & \sum_k \mu(Q_k) \sup_{Q_k \in \mathcal{Q}_0} \sup_{x \in Q_k} \left| g(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} g(y) d\mu(y) \right|^2 \\ = \ & \mu(E) \sup_{Q_k \in \mathcal{Q}_0} \sup_{x \in Q_k} \left| g(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} g(y) d\mu(y) \right|^2, \end{aligned}$$

which goes to zero as ϵ goes to zero. Therefore, from (3.1.0.7), e_ϵ goes to zero as ϵ goes to zero. Letting ϵ to 0, we have $r_1/L \leq \delta_0 \to 0$. Thus (3.1.0.6) becomes

$$\int_E |f(x)|^2 d\mu(x) = \int_E (f(x))^2 d\mu(x) \leq C \liminf_{L \to \infty} L^{n+\alpha} \int_{E(r/L)} |\phi_L * f d\mu(x)|^2 dx, \quad (3.1.0.8)$$

$$\begin{aligned} \int_E |f(x)|^2 d\mu(x) & \leq C \liminf_{L \to \infty} L^{n+\alpha} \int_{E(r/L)} |\phi_L * f d\mu(x)|^2 dx \\ & \leq C \liminf_{L \to \infty} L^{n+\alpha} \int_{\mathbb{R}^n} |\widehat{\phi_L * f d}\mu(\xi)|^2 d\xi \\ & \leq C \liminf_{L \to \infty} L^{-n+\alpha} \int_{\mathbb{R}^n} |\widehat{\phi}(\xi/L)|^2 |\widehat{f d\mu}(\xi)|^2 d\xi. \end{aligned}$$

Since the support of $\widehat{\phi}$ is in the unit ball, we have

$$\int_E |f(x)|^2 d\mu(x) \leq C \|\phi\|^2_{L^1(\mathbb{R}^n)} \liminf_{L \to \infty} \frac{1}{L^{n-\alpha}} \int_{B_L(0)} |\widehat{f d\mu}(\xi)|^2 d\xi.$$

Applying Holder's inequality,

$$\int_E |f(x)|^2 d\mu(x) \leq C \liminf_{L \to \infty} \left(\frac{1}{L^{n-\alpha p/2}} \int_{B_L(0)} |\widehat{f d\mu}(\xi)|^p d\xi \right)^{2/p}.$$

The assumption on the support of $\widehat{\phi}$ to be in the unit ball is used only in the last step. Consider $\phi(x) = e^{-\frac{|x|^2}{2}}$. Proceeding in a similar way, we have

$$\begin{aligned} \int_E |f(x)|^2 d\mu(x) & \leq C \liminf_{L \to \infty} \frac{1}{L^{n-\alpha}} \int_{\mathbb{R}^n} e^{-\frac{|\xi|^2}{2L^2}} |\widehat{f d\mu}(\xi)|^2 d\xi \\ & \leq \tilde{C} \liminf_{L \to \infty} \left(\frac{1}{L^{n-\alpha p/2}} \int_{\mathbb{R}^n} e^{-\frac{|\xi|^2}{2L^2}} |\widehat{f d\mu}(\xi)|^p d\xi \right)^{2/p}. \end{aligned}$$

Hence the proof. $\square$

Now, we give an analogue result of the Corollary 3.1.5 to any tempered distribution supported in a set E of finite α-dimensional packing measure. We closely follow the arguments in [1] (also see page 174 of [16]). We start with the following lemma.

Lemma 3.1.7. *Let u be a tempered distribution supported in a compact set E. Let χ be a radial C_c^∞ function supported in the unit ball and $\int_{\mathbb{R}^n} \chi(x)dx = 1$. Denote $\chi_\epsilon(x) = \epsilon^{-n}\chi(x/\epsilon)$ and $u_\epsilon = u * \chi_\epsilon$. Let $\sigma_u(r) = \int_{S^{n-1}} |\widehat{u}(r\omega)|^2 d\omega$. Then,*

$$\|u_\epsilon\|^2 \leq C\, \epsilon^{(\alpha-n)(1-\frac{1}{q})} \left(\sup_{\epsilon L>1} \frac{1}{L^k} \int_0^L (\sigma_u(r))^{\frac{p}{2}} r^{n-1} dr \right)^{\frac{2}{p}},$$

for some non-zero finite constants C independent of ϵ and $k = n - \frac{\alpha p}{2} - (n-\alpha)\frac{p}{2q}$ with $1 < q \leq \infty$ and $2 \leq p < 2n/\alpha$.

Proof. By the Plancherel theorem,

$$\begin{aligned}
\|u_\epsilon\|^2 &= \int_{\mathbb{R}^n} |\widehat{u}(\xi)|^2 |\widehat{\chi}(\epsilon\xi)|^2 d\xi \\
&= \int_0^{\epsilon^{-1}} (\sigma_u(r))|\widehat{\chi}(\epsilon r)|^2 r^{n-1} dr \;+ \sum_{j=1}^{\infty} \int_{2^{j-1}\epsilon^{-1}}^{2^j\epsilon^{-1}} (\sigma_u(r))|\widehat{\chi}(\epsilon r)|^2 r^{n-1} dr. \\
&\leq \left(\int_0^{\epsilon^{-1}} (\sigma_u(r))^{p/2} r^{n-1} dr \right)^{\frac{2}{p}} \left(\epsilon^{-n} \int_0^1 |\hat{\chi}(r)|^{\frac{2}{1-\frac{2}{p}}} r^{n-1} dr \right)^{1-\frac{2}{p}} \\
&\quad + \sum_{j=1}^{\infty} \left(\int_{\frac{2^{j-1}}{\epsilon}}^{\frac{2^j}{\epsilon}} \sigma_u(r)^{p/2} r^{n-1} dr \right)^{\frac{2}{p}} \left(\epsilon^{-n} \int_{2^{j-1}}^{2^j} |\hat{\chi}(r)|^{\frac{2}{1-\frac{2}{p}}} r^{n-1} dr \right)^{1-\frac{2}{p}} \\
&\leq \epsilon^{(\alpha-n)(1-\frac{1}{q})} \left(\sum_{j=0}^{\infty} a_j \left(\sup_{\epsilon L>1} \frac{1}{L^k} \int_0^L \sigma_u(r)^{\frac{p}{2}} r^{n-1} dr \right)^{\frac{2}{p}} \right),
\end{aligned}$$

where, for all $j > 0$

$$a_j = \left(2^{\frac{2kj}{p-2}} \int_{2^{j-1}}^{2^j} |\widehat{\chi}(r)|^{\frac{2p}{p-2}} r^{n-1} dr \right)^{1-\frac{2}{p}}$$

and $a_0 = \left(\int_0^1 |\widehat{\chi}(r)|^{\frac{2p}{p-2}} r^{n-1} dr \right)^{1-\frac{2}{p}}$. We have $\sum_j a_j$ is finite. Thus

$$\|u_\epsilon\|^2 \leq \epsilon^{(\alpha-n)(1-\frac{1}{q})} C \left(\sup_{\epsilon L>1} \frac{1}{L^k} \int_L^{2L} \sigma_u(r)^{\frac{p}{2}} r^{n-1} dr \right)^{\frac{2}{p}}$$

since $k = n - \frac{\alpha p}{2} - (n-\alpha)\frac{p}{2q}$. □

Theorem 3.1.8. *Fix $0 < \alpha < n$. Let M be a compact set such that $\mathcal{P}^\alpha(M) < \infty$. Let u be a tempered distribution such that support of u is contained in M and $\sigma_u(r) = \int_{S^{n-1}} |\widehat{u}(r\omega)|^2 d\omega$. Let $2 \le p < \frac{2n}{\alpha}$. Then*

$$\|u\|_1^p \le C \limsup_{L\to\infty} \frac{1}{L^{n-\frac{\alpha p}{2}}} \int_0^L (\sigma_u(r))^{\frac{p}{2}} r^{n-1} dr \le C' \limsup_{L\to\infty} \frac{1}{L^{n-\frac{\alpha p}{2}}} \int_{|\xi|<L} |\widehat{u}(\xi)|^p d\xi,$$

where $\|u\|_1 = sup\{< u, \psi >: \psi \in C_c^\infty(\mathbb{R}^n),\ \|\psi\|_{L^\infty(\mathbb{R}^n)} \le 1\}$, C and C' are non zero finite constants depending only on $n,\ \alpha$ and p.

In general, for $2 \le p < \frac{2n}{\alpha + \frac{n-\alpha}{q}}$, where $1 < q \le \infty$

$$\begin{aligned} \|u\|_r^p &\le C \limsup_{L\to\infty} \frac{1}{L^{n-\frac{\alpha p}{2}-(n-\alpha)\frac{p}{2q}}} \int_0^L (\sigma_u(r))^p 2r^{n-1} dr \\ &\le C' \limsup_{L\to\infty} \frac{1}{L^{n-\frac{\alpha p}{2}-(n-\alpha)\frac{p}{2q}}} \int_{|\xi|<L} |\widehat{u}(\xi)|^p d\xi, \end{aligned}$$

where $\frac{1}{r} + \frac{1}{2q} = 1$, $\|u\|_r = sup\{< u, \psi >: \|\psi\|_{L^{2q}(\mathbb{R}^n)} \le 1\}$, C and C' are non zero finite constants depending on $n,\ \alpha$, p and q.

Proof. Choose an even function $\chi \in C_c^\infty(R^n)$ with support in unit ball and $\int_{\mathbb{R}^n} \chi(x)dx = 1$. Let $\chi_\epsilon(x) = \epsilon^{-n}\chi(x/\epsilon)$ and $u_\epsilon = u * \chi_\epsilon$. Then by Lemma 3.1.7,

$$\|u_\epsilon\|^2 \leqslant C\, \epsilon^{(\alpha-n)(1-\frac{1}{q})} \left(\sup_{\epsilon L>1} \frac{1}{L^k} \int_L^{2L} (\sigma_u(r))^{\frac{p}{2}} r^{n-1} dr \right)^{\frac{2}{p}}.$$

Let $\psi \in C_c^\infty(\mathbb{R}^n)$. Let $S = supp\ u \cap supp\ \psi$ where $supp\ \psi$ is contained in a ball $B_{R_\psi}(0)$ of radius R_ψ. Since $supp\ u \subset M$, where M is of finite α-packing measure, S is supported in a set of finite α-pakcing measure. Since S is a bounded subset of M, by Lemma 2.1.1, we have

$$\limsup_{\epsilon\to 0} |S_\epsilon| \epsilon^{\alpha-n} \le c\mathcal{P}^\alpha(S) < \infty.$$

For given $0 < \delta < 1$, there exists ϵ_0 such that for all $\epsilon < \epsilon_0$, $|S(\epsilon)|\epsilon^{\alpha-n} \le C(\mathcal{P}^\alpha(M) + \delta) \le C_M$.

So, for $k = n - \frac{\alpha p}{2} - (n-\alpha)\frac{p}{2q}$,

$$\begin{aligned}
|<u_\epsilon, \psi>|^2 &\leq \|u_\epsilon\|_2^2 \int_{S_\epsilon} |\psi|^2 \\
&\leq \|u_\epsilon\|_2^2 \Big(\int_{\mathbb{R}^n} |\psi|^{2q} \Big)^{\frac{1}{q}} |S_\epsilon|^{1-\frac{1}{q}} \\
&\leq C_M \|\psi\|_{2q}^2 \, \epsilon^{(n-\alpha)(1-\frac{1}{q})} \|u_\epsilon\|_2^2 \\
&\leq C\|\psi\|_{2q}^2 \big(\sup_{\epsilon L > 1} \frac{1}{L^k} \int_0^L (\sigma_u(r))^{\frac{p}{2}} r^{n-1} dr \big)^{\frac{2}{p}}.
\end{aligned}$$

Thus

$$\begin{aligned}
\|u\|_r^p &\leq C \limsup_{L\to\infty} \frac{1}{L^{n-\frac{\alpha p}{2}-(n-\alpha)\frac{p}{2q}}} \int_0^L (\sigma_u(r))^p 2r^{n-1} dr \\
&\leq C' \limsup_{L\to\infty} \frac{1}{L^{n-\frac{\alpha p}{2}-(n-\alpha)\frac{p}{2q}}} \int_{|\xi|<L} |\widehat{u}(\xi)|^p d\xi.
\end{aligned}$$

□

In [1], the authors proved the following:

Theorem 3.1.9. *Let u be a tempered distribution such that $\widehat{u} \in L^2_{loc}$ and*

$$\limsup_{L\to\infty} \frac{1}{L^k} \int_{|\xi|\leq L} |\widehat{u}(\xi)|^2 d\xi < \infty.$$

If the restriction of u to an open subset X of $\mathbb{R}^n$ is supported by a C^1-submanifold M of codimension k, then it is an L^2-density $u_0 dS$ on M and

$$\int_M |u_0|^2 dS \leq C \limsup_{L\to\infty} \frac{1}{L^k} \int_{|\xi|\leq R} |\widehat{u}(\xi)|^2 d\xi,$$

where C only depends on n.

We prove an analogue of the above theorem for fractional dimensional sets.

Theorem 3.1.10. *Let u be a tempered distribution supported in a set E of finite α-packing measure such that for some $2 \leq p < 2n/\alpha$,*

$$\limsup_{L\to\infty} \frac{1}{L^{n-\frac{\alpha p}{2}}} \int_{|\xi|\leq L} |\widehat{u}(\xi)|^p d\xi < \infty.$$

Then u is an L^2 density $u_0 \, d\mathcal{P}^\alpha$ on E and

$$\Big(\int_E |u_0|^2 d\mathcal{P}^\alpha \Big)^{p/2} \leq C \limsup_{L\to\infty} \frac{1}{L^{n-\frac{\alpha p}{2}}} \int_{|\xi|\leq L} |\widehat{u}(\xi)|^p d\xi < \infty.$$

Proof. Let $\psi \in C_c^\infty(\mathbb{R}^n)$. Let $S = supp\, u \cap supp\, \psi$. Then S is bounded and let M be the smallest closed cube that contains S. As in Theorem 3.1.6, for $0 < \delta < 1$, let $\tilde{\mathcal{Q}}_0$ be the collection of all half open cubes $Q_k = \{x = (x_1, ...x_n) \in M : (k_j - 1)\delta < x_j \leq k_j\delta\}$, $(k = (k_1, ...k_n),\ k_j \in \mathbb{Z})$ and $\mathcal{Q}_0$ be the collection of all $Q_k \in \tilde{\mathcal{Q}}_0$ such that $\mathcal{P}^\alpha(Q_k \cap E) \neq 0$. Denote $\mu = \mathcal{P}^\alpha|_S$. $\mathcal{P}^\alpha(S) \leq \mathcal{P}^\alpha(E) < \infty$ implies μ is Radon. Since S is bounded, there are finite Q_k's in $\mathcal{Q}_0$. Let $\delta_0 = \min_{Q_k \in \mathcal{Q}_0}\{\mu(Q_k)\}$. By Lemma 2.1.1, for each k, there exists δ_k such that

$$\begin{aligned} |(Q_k \cap S)(\epsilon)|\epsilon^{\alpha-n} &\leq C_n\mathcal{P}^\alpha(Q_k \cap S) + C_n\tilde{\delta_0}\delta \\ &\leq 2C_n\mathcal{P}^\alpha(Q_k \cap S) = 2C_n\mu(Q_k), \quad (3.1.0.9) \\ |S(\epsilon)|\epsilon^{\alpha-n} &\leq \mu(S) + \delta \quad (3.1.0.10) \end{aligned}$$

for all $\epsilon \leq \delta_k$. Fix $\epsilon_0 = \min\{\delta,\ \delta_0,\ \delta_1,\ \delta_2, ..\}$. For every $\epsilon < \epsilon_0$, let $\mathcal{Q}_0^\epsilon$ denote the collection of all Q_k in $\tilde{\mathcal{Q}}_0$ such that $|Q_k \cap S(\epsilon)| \neq 0$.

$$\begin{aligned} \epsilon^{\alpha-n}\int_{S(\epsilon)}|\psi(x)|^2dx &= \epsilon^{\alpha-n}\sum_{Q_k\in\mathcal{Q}_0^\epsilon}\int_{Q_k\cap S(\epsilon)}|\psi(x)|^2dx \\ &\leq \epsilon^{\alpha-n}\sum_{Q_k\in\mathcal{Q}_0^\epsilon\backslash\mathcal{Q}_0}\int_{Q_k\cap S(\epsilon)}|\psi(x)|^2dx \\ &\quad +2\epsilon^{\alpha-n}\sum_{Q_k\in\mathcal{Q}_0}\int_{Q_k\cap S(\epsilon)}|\psi(x) - \frac{1}{\mu(Q_k)}\int_{Q_k}\psi(y)d\mu(y)|^2dx \\ &\quad +2\epsilon^{\alpha-n}\sum_{Q_k\in\mathcal{Q}_0}\int_{Q_k\cap S(\epsilon)}|\frac{1}{\mu(Q_k)}\int_{Q_k}\psi(y)d\mu(y)|^2dx. \end{aligned}$$

Since, for $Q_k \in \mathcal{Q}_0^\epsilon\backslash\mathcal{Q}_0$, $\mu(Q_k) = 0$, from (3.1.0.9),

$$\begin{aligned} &\epsilon^{\alpha-n}\sum_{Q_k\in\mathcal{Q}_0^\epsilon\backslash\mathcal{Q}_0}\int_{Q_k\cap S(\epsilon)}|\psi(x)|^2dx \\ &\leq 2C_n\|\psi\|_\infty^2\sum_{Q_k\in\mathcal{Q}_0^\epsilon\backslash\mathcal{Q}}\mu(Q_k) = 0. \end{aligned}$$

Hence,

$$\begin{aligned} \epsilon^{\alpha-n}\int_{S(\epsilon)}|\psi(x)|^2dx &\leq 2\epsilon^{\alpha-n}\sum_{Q_k\in\mathcal{Q}_0}\int_{Q_k\cap S(\epsilon)}|\psi(x) - \frac{1}{\mu(Q_k)}\int_{Q_k}\psi(y)d\mu(y)|^2dx \\ &\quad +2\epsilon^{\alpha-n}\sum_{Q_k\in\mathcal{Q}_0}\int_{Q_k\cap S(\epsilon)}|\frac{1}{\mu(Q_k)}\int_{Q_k}\psi(y)d\mu(y)|^2dx \\ &\leq e_\delta + 2\sum_{Q_k\in\mathcal{Q}_0}\epsilon^{\alpha-n}|Q_k \cap S(\epsilon)|\frac{1}{\mu(Q_k)}\int_{Q_k}|\psi(y)|^2d\mu(y), \end{aligned}$$

where

$$e_\delta = 2 \sum_{Q_k \in \mathcal{Q}_0} \epsilon^{\alpha-n} \int_{Q_k \cap S(\epsilon)} |\psi(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} \psi(y) d\mu(y)|^2 dx.$$

By (3.1.0.9),

$$\epsilon^{\alpha-n} |Q_k \cap S(\epsilon)| \le \epsilon^{\alpha-n} |(Q_k \cap S)(\epsilon)| \le 2C_n \mu(Q_k).$$

Hence

$$\begin{aligned} \epsilon^{\alpha-n} \int_{S(\epsilon)} |\psi(x)|^2 dx &\le e_\delta + 4C_n \sum_{Q_k \in \mathcal{Q}_0} \int_{Q_k} |\psi(y)|^2 d\mu(y) \\ &= e_\delta + 4C_n \int_E |\psi(y)|^2 d\mu(y). \end{aligned} \tag{3.1.0.11}$$

Since ψ is compactly supported continuous function, $|\psi(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} \psi(y) d\mu(y)| \to 0$ uniformly in x and Q_k as δ goes to zero. $\sup\limits_{x \in S(\epsilon)} |\psi(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} \psi(y) d\mu(y)| \to 0$ as δ goes to zero.

$$\begin{aligned} e_\delta &= \epsilon^{\alpha-n} \sum_{Q_k \in \mathcal{Q}_0} \int_{Q_k \cap S(\epsilon)} |\psi(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} \psi(y) d\mu(y)|^2 dx \\ &\le \epsilon^{\alpha-n} \sum_{Q_k \in \mathcal{Q}_0} |Q_k \cap S(\epsilon)| \sup_{x \in S(\epsilon)} |\psi(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} \psi(y) d\mu(y)|^2 \\ &\le \epsilon^{\alpha-n} |S(\epsilon)| \sup_{x \in S(\epsilon)} |\psi(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} \psi(y) d\mu(y)|^2. \end{aligned}$$

Then together with (3.1.0.10), e_δ goes to zero as δ goes to zero. Thus from (3.1.0.11), for given $0 < \delta < 1$, there exists small ϵ_0 such that for all $\epsilon < \epsilon_0$,

$$\begin{aligned} \epsilon^{\alpha-n} \int_{S(\epsilon)} |\psi(x)|^2 dx &\le e_\delta + 4C_n \int_E |\psi(y)|^2 d\mathcal{P}^\alpha(y) \\ &= e_\delta + 4C_n \|\psi\|^2_{L^2(d\mathcal{P}^\alpha|_E)}. \end{aligned} \tag{3.1.0.12}$$

where e_δ tends to zero as δ tends to zero.

Now we proceed as in the Theorem 3.1.8. Choose an even function $\chi \in C_c^\infty(R^n)$ with support in unit ball and $\int_{\mathbb{R}^n} \chi(x)dx = 1$. Let $\chi_\epsilon(x) = \epsilon^{-n}\chi(x/\epsilon)$ and $u_\epsilon = u * \chi_\epsilon$. Then by Lemma 3.1.7,

$$\begin{aligned} \|u_\epsilon\|^2 &\le C\, \epsilon^{\alpha-n} \Big(\sup_{\epsilon L > 1} \frac{1}{L^{n-\alpha p/2}} \int_0^L (\sigma_u(r))^{\frac{p}{2}} r^{n-1} dr \Big)^{\frac{2}{p}} \\ &\le C\, \epsilon^{\alpha-n} \Big(\sup_{\epsilon L > 1} \frac{1}{L^{n-\alpha p/2}} \int_{|\xi| < L} |\hat{u}(\xi)|^p d\xi \Big)^{\frac{2}{p}}. \end{aligned} \tag{3.1.0.13}$$

We have $\epsilon \to 0$ as $\delta \to 0$. Thus

$$\begin{aligned} |<u,\psi>|^2 &= \lim_{\epsilon\to 0} |<u_\epsilon,\psi>|^2 \\ &\leq \lim_{\epsilon\to 0} \|u_\epsilon\|_2^2 \int_{S_\epsilon} |\psi|^2 \\ &\leq \lim_{\epsilon\to 0} \|u_\epsilon\|_2^2 \epsilon^{n-\alpha}(e_\delta + C\|\psi\|^2_{L^2(d\mathcal{P}^\alpha|_E)}) \text{ from (3.1.0.12).} \end{aligned}$$

Thus letting δ go to zero, together with (3.1.0.13),

$$|<u,\psi>|^2 \leq C\|\psi\|^2_{L^2(d\mathcal{P}^\alpha|_E)} \left(\limsup_{L\to\infty} \frac{1}{L^{n-\alpha p/2}} \int_{|\xi|\leq L} |\hat{u}(\xi)|^p d\xi\right)^{\frac{2}{p}}$$

Thus u is an L^2 density $u_0\, d\mathcal{P}^\alpha$ on E and

$$\left(\int_E |u_0|^2 d\mathcal{P}^\alpha\right)^{p/2} \leq C \limsup_{L\to\infty} \frac{1}{L^{n-\frac{\alpha p}{2}}} \int_{|\xi|\leq L} |\widehat{u}(\xi)|^p d\xi < \infty.$$

□

3.2 L^p-Fourier asymptotic properties of fractal measures for $1 \leq p < 2$

Let μ denote a fractal measure supported in an α-dimensional set $E \subset \mathbb{R}^n$ and $f \in L^q(d\mu)$ ($1 \leq q \leq \infty$). Suppose $1 \leq p \leq 2$ dependent on q. In this section, we obtain upper and lower bounds for

$$L^{-k} \int_{|\xi|\leq L} |\widehat{fd\mu}(\xi)|^p d\xi,$$

for very large L and positive k dependent on $\alpha,\ p$ and n.

Let $\tilde{\psi}_t(x) = t^{-n}\tilde{\psi}(t^{-1}x)$, where $|\tilde{\psi}(x)| \leq \psi(|x|)$, ψ is decreasing, bounded and $\int_0^\infty \psi(r) r^{n-1} dr < \infty$. Let $u_t(x) = \tilde{\psi}_t * (fd\mu)(x) = \int \tilde{\psi}_t(x-y) f(y) d\mu(y)$. Then Strichartz in [41] proved the following:

Theorem 3.2.1. *[41]: Let $\mu = \mathcal{H}_\alpha|_E$. If $f \in L^p(d\mu)$, ($1 \leq p \leq \infty$), then*

1. *If E is locally uniformly α-dimensional, for $0 < t \leq 1$,*

$$\left(\int |u_t(x)|^p\, dx\right)^{1/p} \leq ct^{(\alpha-n)/p'} \|f\|_{L^p(d\mu)}.$$

2. *If E is only quasi α-regular, then*

$$\liminf_{t\to 0} t^{n-\alpha}|u_t(x)| \geq c|f(x)|$$

for $\mathcal{H}_\alpha$-almost every x in E.

Theorem 3.2.2. *Let $f \in L^\infty(d\mathcal{P}^\alpha)$ be supported in an quasi α-regular set $E \subset \mathbb{R}^n$ for some $0 \leq \alpha \leq n$. Then*

$$\|f\|_{L^\infty(d\mu)} \leq c \liminf_{L\to\infty} \frac{1}{L^{n-\alpha}} \int_{\mathbb{R}^n} e^{-\frac{|\xi|^2}{2L^2}} |\widehat{fd\mu}(\xi)| d\xi, \tag{3.2.0.1}$$

where c is a constant independent of f and $d\mu = d\mathcal{H}_\alpha|_E$.

Proof. By Theorem 3.2.1, we have

$$\liminf_{t\to 0} t^{n-\alpha}|u_t(x)| \geq c|f(x)| \text{ a.e. } x \in E \tag{3.2.0.2}$$

where $u_t(x) = \tilde{\psi}_t * (fd\mu)(x) = \int \tilde{\psi}_t(x-y)f(y)d\mu(y)$, with $\tilde{\psi}(x) = e^{-\frac{|x|^2}{t}}$.

$$\begin{aligned}
|f(x)| &\leq c \liminf_{t\to 0} t^{n-\alpha}|u_t(x)| \\
&= c \liminf_{t\to 0} t^{-\alpha}\Big| \int_E e^{-\frac{|x-y|^2}{2t^2}} f(y)d\mu(y)\Big| \\
&= c \liminf_{t\to 0} t^{-\alpha}\Big| \int_E \int_{\mathbb{R}^n} e^{-\frac{|\xi|^2}{2}} e^{i(x-y).\xi/t} d\xi f(y) d\mu(y)\Big| \\
&= c \liminf_{t\to 0} t^{n-\alpha}\Big| \int_{\mathbb{R}^n} e^{-\frac{(t|\xi|)^2}{2}} \widehat{fd\mu}(\xi) e^{ix.\xi} d\xi\Big| \\
&\leq c \liminf_{t\to 0} t^{n-\alpha} \int_{\mathbb{R}^n} e^{-\frac{(t|\xi|)^2}{2}} |\widehat{fd\mu}(\xi)| d\xi.
\end{aligned}$$

Hence from (3.2.0.2), substituting $L = t^{-1}$ in the above equation, we get (3.2.0.1). Hence the proof. $\square$

Strichartz proved the following analogue of the Hausdorff-Young inequality in [41].

Theorem 3.2.3. *[41] If $f \in L^{p'}(d\mu)$ for $2 \leq p \leq \infty$ and μ is locally uniformly α-dimensional then*

$$\sup_x \sup_{L\geq 1} \frac{1}{L^{n-\alpha}} \int_{B_L(x)} |\widehat{fd\mu}(\xi)|^p d\xi \leq c\|f\|_{p'}^p,$$

where $1/p + 1/p' = 1$ for $2 \leq p < \infty$ and for $p = \infty$,

$$\|\widehat{fd\mu}\|_{L^\infty(\mathbb{R}^n)} \leq c\,\|f\|_{L^1(d\mu)}.$$

Applying Holder's inequality in Theorem 3.2.3, we obtain the following:

Corollary 3.2.4. *Let $d\mu = \phi d\mathcal{H}_\alpha + \nu$ (as in the Remark 3.1.2) be a locally uniformly α-dimensional measure on $\mathbb{R}^n$. For any $f \in L^q(d\mu)$ ($1 \le p \le q' \le 2 \le q$,) supported in a finite μ-measurable set E, we have for a fixed y and a constant c independent of y,*

$$\limsup_{L\to\infty} \frac{1}{L^{n-\alpha p/q}} \int_{B_L(y)} |\widehat{(fd\mu)}|^p \le c \ \Big(\int_E |f(x)|^q \phi(x) d\mathcal{H}_\alpha(x) \Big)^{p/q}.$$

Proof.

$$\begin{aligned} & \limsup_{L\to\infty} L^{\frac{\alpha p}{q}-n} \int_{B_L(y)} |\widehat{(fd\mu)}|^p \\ \le\ & \limsup_{L\to\infty} (L^{\alpha-n} \int_{B_r(y)} |\widehat{(fd\mu)}|^{q'})^{p/q'} \\ \le\ & c\Big(\int_E |f(x)|^q \phi(x) d\mathcal{H}_\alpha(x) \Big)^{p/q}. \end{aligned}$$

$\square$

In a different direction, the authors in [17] proved generalized Hardy inequality for discrete measures:

Theorem 3.2.5. *[17] Let c_k be a sequence of complex numbers, a_k be a sequence of real numbers and $fd\mu_0$ denote the zero dimensional measure $f(x) = \sum_1^\infty c_k \delta(x - a_k)$ where δ is the usual Dirac measure at zero.*

1. *Let $a_1 < a_2 < \ldots$ and assume $\widehat{fd\mu_0} = \sum c_k e^{ia_k x}$ belongs to the class of almost periodic functions. Then,*
$$\sum_1^\infty \frac{|c_k|}{k} \le C \lim_{L\to\infty} L^{-1} \int_{-L}^{L} |\widehat{fd\mu_0}(x)| dx.$$

2. *Let a_k be a sequence of real numbers, not necessarily increasing and $1 < p \le 2$. Assume that $u(x) = \widehat{fd\mu_0(x)}$ converges to $\sum_1^\infty c_k e^{ia_k x}$ in the class of almost periodic functions. Then*
$$\sum_1^\infty \frac{|c_k|^p}{k^{2-p}} \le \sum_1^\infty \frac{|c_k'|^p}{k^{2-p}} \le C \ \lim L^{-1} \int_{-L}^{L} |u(x)|^p dx,$$
where c_k' is the nonincreasing rearrangement of the sequence $|c_k|$.

The authors also proved generalized Hardy inequality for fractal measures $fd\mu$ on $\mathbb{R}^1$ of dimension α $(0 < \alpha < 1)$ in [17] by generalizing part (1) of the above theorem with additional hypothesis on μ. To prove the same, they introduced α-coherent sets in $\mathbb{R}$ $(0 < \alpha < 1)$. Given $x \in \mathbb{R}$ and a set $E \subset \mathbb{R}$, let $E_x = E \cap (-\infty, x]$. Let $s = sup\{x : \mathcal{H}_\alpha(E_x) < \infty\}$, $E^0 = (E_s)^*$ where, for a set E,

$$E^* = \{x \in E : 2^{-\alpha} \leq \overline{D^\alpha}(\mathcal{H}_\alpha|_E, x) \leq 1\}.$$

The set $E \subset \mathbb{R}$ is α-coherent $(0 < \alpha < 1)$, if there is a constant C such that for all $x \leq s$,

$$\limsup_{\delta \to 0} |E_x^0(\delta)|\delta^{\alpha-1} \leq C\mathcal{H}_\alpha(E_x^0),$$

where $|E_x^0(\delta)|$ denotes the one dimensional Lebesgue measure of the δ-distance set $E_x^0(\delta)$ of E_x^0. The following was proved in [17].

Theorem 3.2.6. *[17] Suppose $0 < \alpha < 1$, $f \in L^1(d\mathcal{H}_\alpha)$ and $\mu = \mathcal{H}_\alpha|_E$ where E is either α-coherent or quasi α-regular. Then, there exists a non-zero finite constant independent of f such that*

$$\int_E \frac{|f(x)|d\mu(x)}{\mathcal{H}_\alpha(E_x^0)} \leq \ C \liminf_{L \to \infty} L^{\alpha-1} \int_{-L}^{L} |\widehat{fd\mu}(x)|dx.$$

Remark 3.2.7. *Examples in [17] show that there are quasi regular sets in $\mathbb{R}$ which are not α-coherent and there are α-coherent sets which are not quasi regular, for given $0 < \alpha < 1$.*

In this section, using the packing measure and finding a continuous analogue of the arguments used in the proof of the Theorem 3.2.6, we prove an analogue version of part(2) of the Theorem 3.2.5 for $0 < \alpha < n$, $n \geq 1$ and $1 \leq p \leq 2$ with a slight modification in the hypothesis:

Theorem 3.2.8. *Let $E \subset \mathbb{R}^n$ be a compact set of finite α-dimensional packing measure and $\mu = \mathcal{P}^\alpha|_E$. Let $f \in L^p(d\mu)$ be a positive function, for $1 \leq p \leq 2$. Then there exists a constant C independent of f such that*

$$\int_E \frac{|f(x)|^p}{[\mu(E_x)]^{2-p}} d\mu(x) \leq \ C \liminf_{L \to \infty} \frac{1}{L^{n-\alpha}} \int_{|\xi| \leq L} |\widehat{fd\mu}(\xi)|^p d\xi, \qquad (3.2.0.3)$$

where $E_x = E \cap [(-\infty, x_1] \times ... \times (-\infty, x_n]]$ for $x = (x_1, ...x_n) \in \mathbb{R}^n$.

First we prove the following lemma:

Lemma 3.2.9. *Suppose $L > 1$ and $0 < \delta = r/L < 1$ are given constants. Let $g_L \in L^1(\mathbb{R}^n)$ and $S_\delta = \cup_{i=1}^s \Delta_i^\delta$ be the union of disjoint cubes such that $0 < |\Delta_i^\delta| < \delta^n$. Then, there exists a non-zero finite constant C_2 independent of g_L, s, δ and L such that*

$$\frac{\delta^{-n}}{P_\delta}\int_{S_\delta}|g_L(x)|dx \leq C_2 \int_{\mathbb{R}^n}|\widehat{g_L}(\xi)|d\xi, \tag{3.2.0.4}$$

where $P_\delta > 1$ is a constant dependent on δ.

Proof. For all $i = 1, ...s$, construct $f_i \in L^2(\mathbb{R}^n)$ such that

$$\begin{aligned} |\widehat{f}_i(x)| &= \frac{\delta^{-n}}{P_\delta} \text{ for } x \in \Delta_i^\delta \\ &= 0 \text{ for } x \notin \Delta_i^\delta \\ \widehat{f}_i(x)g_L(x) &\geq 0. \end{aligned}$$

Since $|\Delta_i^\delta| \leq \delta^n$ and $P_\delta > 1$, $\|\widehat{f}_i\|_1 \leq 1$ and hence for all ξ, $|f_i(\xi)| \leq 1$. Denote $F_0 \equiv 0$. For all $i = 1, ..., s$, let

$$F_i(\xi) = \frac{4}{5}F_{i-1}(\xi)\exp(\frac{-1}{4s^2}|f_i(\xi)|) + \frac{f_i(\xi)}{20}$$

and denote $F \equiv F_s$. Since $|f_i(\xi)| \leq 1$ for all i, we have $|F_1(\xi)| \leq 1/4$. Note that for all $0 \leq t \leq 1$ and $s \geq 1$,

$$\begin{aligned} \frac{4}{5}exp(\frac{-t}{4s^2}) &\leq 1 - \frac{t}{5} \\ \frac{1}{5}exp(\frac{-t}{4s^2}) + \frac{t}{20} &\leq \frac{1}{4}. \end{aligned}$$

Since for all ξ, $|f_2(\xi)| \leq 1$, we have

$$|F_2(\xi)| \leq \frac{1}{5}exp(\frac{-|f_2(\xi)|}{4s^2}) + \frac{|f_2(\xi)|}{20} \leq \frac{1}{4}.$$

Then by induction $\|F\|_\infty \leq 1/4$. By construction, we have

$$F(\xi) = \sum_{k=1}^{s-1}\left[\frac{4^{s-k}f_k(\xi)}{5^{s-k}20}\exp\big(\frac{-1}{4s^2}\sum_{l=k+1}^{s}|f_l(\xi)|\big)\right] + \frac{f_s(\xi)}{20}. \tag{3.2.0.5}$$

Now consider $\widehat{F}$,

$$\begin{aligned}\widehat{F}(x) &= \sum_{k=1}^{s-1}\Big[\frac{4^{s-k}f_k(\xi)}{5^{s-k}20}\exp\big(\frac{-1}{4s^2}\sum_{l=k+1}^{s}|f_l(\xi)|\big)\Big]\widehat{\ }(x)+\frac{\widehat{f_s}(x)}{20}\\ &= \sum_{k=1}^{s-1}\Big[\frac{4^{s-k}f_k(\xi)}{5^{s-k}20}\big(\exp(\frac{-1}{4s^2}\sum_{l=k+1}^{s}|f_l(\xi)|)-1\big)\Big]\widehat{\ }(x)\\ &\quad+\sum_{k=1}^{s}\frac{4^{s-k}\widehat{f_k}(x)}{5^{s-k}20}.\end{aligned}$$

By the construction of $f'_{i_0}s$, for all $x\in\Delta^{\delta}_{i_0}$, $|\widehat{f_i}(x)|=0$ for all $i\neq i_0$ and $\widehat{f_i}(x)g_L(x)\geq 0$. Hence

$$\begin{aligned}&Re(\widehat{F}(x)g_L(x))\\ &\leq \sum_{k=1}^{s-1}\Big|\Big[\frac{4^{s-k}f_k(\xi)}{5^{s-k}20}\big(\exp(\frac{-1}{4s^2}\sum_{l=k+1}^{s}|f_l(\xi)|)-1\big)\Big]\widehat{\ }(x)\Big||g_L(x)|\\ &\quad+\frac{4^{s-i_0}}{5^{s-i_0}20}\widehat{f_{i_0}}(x)g_L(x)\\ &\leq \sum_{k=1}^{s-1}\frac{\|f_k\|_2}{20}\Big\|\big(\exp(\frac{-1}{4s^2}\sum_{l=k+1}^{s}|f_l(\xi)|)-1\big)\widehat{\ }\Big\|_2|g_L(x)|\\ &\quad+\frac{1}{20}\widehat{f_{i_0}}(x)g_L(x).\end{aligned}$$

That is, for $x\in\Delta^{\delta}_{i_0}$,

$$\begin{aligned}&Re(20\widehat{F}(x)g_L(x))-\widehat{f_{i_0}}(x)g_L(x)\\ &\leq \sum_{k=1}^{s-1}\|f_k\|_2\Big\|\big(\exp(\frac{-1}{4s^2}\sum_{l=k+1}^{s}|f_l(\xi)|)-1\big)\widehat{\ }\Big\|_2|g_L(x)|.\end{aligned}\tag{3.2.0.6}$$

Since for all $a>0$, $\left|\frac{\exp(-a)+1}{a}\right|\leq 1$ and for all i, $\|f_i\|_2\leq\delta^{-n/2}$ we have

$$\begin{aligned}\sum_{k=1}^{s-1}\|f_k\|_2\Big\|\big(\exp(\frac{-1}{4s^2}\sum_{l=k+1}^{s}|f_l(\xi)|)-1\big)\widehat{\ }\Big\|_2 &\leq \sum_{k=1}^{s-1}\|f_k\|_2\Big(\sum_{l=k+1}^{s}\frac{\|f_l\|_2}{4s^2}\Big)\\ &\leq \frac{\delta^{-n}}{8}.\end{aligned}$$

Thus from (3.2.0.6), for $x\in\Delta^{\delta}_{i_0}$

$$\begin{aligned}&\delta^{-n}|g_L(x)|=|\widehat{f_{i_0}}(x)g_L(x)|\\ \leq\;& |\widehat{f_{i_0}}(x)g_L(x)-Re(20\widehat{F}(x)g_L(x))|+Re(20\widehat{F}(x)g_L(x))\\ \leq\;& \frac{\delta^{-n}}{8}|g_L(x)|+Re(20\widehat{F}(x)g_L(x)).\end{aligned}$$

Thus for all i and $x \in \Delta_{i_0}^{\delta}$, $0 \leq \delta^{-n}|g_L(x)| \leq 40Re(\widehat{F}(x)g_L(x))$. Hence, for all x, $0 \leq \delta^{-n}|g_L(x)| \leq 40Re(\widehat{F}(x)g_L(x))$ and

$$\begin{aligned}\int_{S_\delta} \delta^{-n}|g_L(x)|dx &\leq 40Re\Big(\int_{\mathbb{R}^n} \widehat{F}(x)g_L(x)dx\Big) \\ &\leq 40\int_{\mathbb{R}^n} |F(\xi)||\widehat{g_L}(\xi)|d\xi,\end{aligned}$$

Also we have $\|F\|_\infty \leq 1/4$. Then,

$$\int_{S_\delta} \delta^{-n}|g_L(x)|dx \leq C_2 \int_{\mathbb{R}^n} |\widehat{g_L}(\xi)|d\xi.$$

Hence the proof. □

Proof of Theorem 3.2.8:

Since E is a bounded set, without loss of generality we assume that $\tilde{m} > 1$ is the smallest integer such that for all $x = (x_1, ...x_n) \in E$, $1 \leq x_j \leq \tilde{m}$, $j = 1, ...n$. Fix $0 < \epsilon < 1$ and $m = \tilde{m} + 1$. Then $E(\epsilon)$, the ϵ-distance set of E is contained in $M = (0, m) \times ...(0, m)$.

As in Theorem 3.1.6, we approximate $f d\mu$ with a Schwartz function on a fine decomposition of $E(r/L)$, r/L-distance set of E for very small r/L depending on ϵ. First, we construct a set C_ϵ as in the proof of Theorem 3.2.6 in [17] such that C_ϵ has small α-packing measure.

Construct a self-similar Cantor-type set C in $[-2/\epsilon, -1/\epsilon] \times ...[-2/\epsilon, -1/\epsilon] \subset \mathbb{R}^n$ satisfying open set condition with dilation factor $0 < \eta < 1$ such that $N\eta^\alpha = 1$ and $\mathcal{H}_\alpha(C) = 1$. (See Definition 1.1.6 in Chapter 1.) Let C_ϵ denote the ϵ-dilated C such that $C_\epsilon \subset [-2, -1] \times .. \times [-2, -1] = M_1$ and $\mathcal{H}_\alpha(C_\epsilon) = \epsilon^\alpha \mathcal{H}_\alpha(C) = \epsilon^\alpha$. By Theorem 1.1.7, C_ϵ is α-regular and $\eta^\alpha \leq \frac{\mathcal{H}_\alpha(C_\epsilon \cap B_r(x))}{r^\alpha}$ for all $0 < r \leq 1$. Then, by the definition of packing measure and part(4) in Lemma 1.1.12, $\epsilon^\alpha = \mathcal{H}_\alpha(C_\epsilon) < \mathcal{P}^\alpha(C_\epsilon) \leq \eta^{-\alpha}\mathcal{H}_\alpha(C_\epsilon) = (\eta^{-1}\epsilon)^\alpha$. Denote $E' = E \cup C_\epsilon$. Thus for all $x \in E$, $\mu(E'_x) = \mu(E_x) + \mathcal{P}^\alpha(C_\epsilon)$. Hence

$$\begin{aligned}\int_E \frac{|f(x)|^p}{\mu(E_x)^{2-p}} d\mu(x) &= \lim_{\epsilon\to 0} \int_E \frac{|f(x)|^p}{(\mu(E_x) + (\eta^{-1}\epsilon)^\alpha + \epsilon)^{2-p}} d\mu(x) \\ &\leq \lim_{\epsilon\to 0} \int_E \frac{|f(x)|^p}{(\mu(E'_x) + \epsilon)^{2-p}} d\mu(x) \qquad (3.2.0.7)\end{aligned}$$

Now to approximate $fd\mu$ with a Schwartz function, we proceed as in the Theorem 3.1.6.

Fix $\epsilon_1 < \epsilon/2$. For each $k = (k_1, ...k_n)$, $(0 < k_j \in \mathbb{Z})$ denote $Q_k = \{x = (x_1, ...x_n) \in M : (k_j - 1)\epsilon_1 < x_j \leq k_j\epsilon_1\}$. Let $\mathcal{Q}_0$ denote the finite collection of all such cubes whose intersection with E that has non zero measure, that is, $\mu(Q_k) \neq 0$. For every $k = (k_1, ...k_n)$, denote $x_k = ((k_1 - 1)\epsilon_1, ...(k_n - 1)\epsilon_1)$, $E_k = E_{x_k} = E \cap \prod_{j=1}^{n}(-\infty, (k_j - 1)\epsilon_1]$ and $E'_k = E'_{x_k} = E' \cap \prod_{j=1}^{n}(-\infty, (k_j - 1)\epsilon_1]$. Then for all $Q_k \in \mathcal{Q}_0$ and $x \in Q_k$, $\mu(E'_k) \leq \mu(E'_x)$. Also for all k, $\mu(E'_k) = \mu(E_k) + \mathcal{P}^\alpha(C_\epsilon) > 0$. Since E is compact, $\mathcal{Q}_0$ has finite disjoint collection of half open cubes. Hence

$$\begin{aligned}
\int_E \frac{|f(x)|^p}{(\mu(E'_x) + \epsilon)^{2-p}} d\mu(x) &= \sum_{Q_k \in \mathcal{Q}_0} \int_{Q_k} \frac{|f(x)|^p}{(\mu(E'_x) + \epsilon)^{2-p}} d\mu(x) \\
&\leq \sum_{Q_k \in \mathcal{Q}_0} \int_{Q_k} \frac{|f(x)|^p}{(\mu(E'_k) + \epsilon)^{2-p}} d\mu(x) \\
&\leq \frac{C_p}{(\epsilon)^{2-p}} \sum_{Q_k \in \mathcal{Q}_0} \int_{Q_k} \left| f(x) - \frac{1}{(\mu(Q_k))^p} \int_{Q_k} f(y) d\mu(y) \right|^p d\mu(x) \\
&\quad + \sum_{Q_k \in \mathcal{Q}_0} \frac{\mu(Q_k)^{1-p}}{(\mu(E'_k) + \epsilon)^{2-p}} \left| \int_{Q_k} f(y) d\mu(y) \right|^p. \qquad (3.2.0.8)
\end{aligned}$$

Let $i_{\epsilon_1} = \inf_{Q \in \mathcal{Q}_0} \mu(Q)$. Since infimum is taken over cubes in $\mathcal{Q}_0$, which is a finite collection and $\mu(Q) \neq 0$, we have $i_{\epsilon_1} > 0$. Now by Lemma 2.1.1, for each k, there exists δ_k such that

$$\begin{aligned}
|(Q_k \cap E)(\delta)|\delta^{\alpha - n} &\leq C_n\mu(Q_k \cap E) + C_n i_{\epsilon_1}\epsilon, \\
&\leq 2C_n\mu(Q_k \cap E) \text{ (since } \epsilon < 1), \qquad (3.2.0.9) \\
|E'_k(\delta)|\delta^{\alpha - n} &\leq C_n\mu(E'_k) + C_n\epsilon, \qquad (3.2.0.10)
\end{aligned}$$

for all $\delta \leq \delta_k$. Let $\tilde{\delta}_1 \leq \min_k\{\delta_k\}$.

Let ϕ be a positive Schwartz function such that $\widehat{\phi}(0) = 1$, support of $\widehat{\phi}$ is supported in the unit ball and there exists $r_1 > 0$ such that

$$\int_{A_{r_1}(0)} \phi(x)dx = 1/2^{n+1}, \qquad (3.2.0.11)$$

where $A_{r_1}(0) = \{x = (x_1, ...x_n) \in \mathbb{R}^n : -r_1 < x_j \leq 0, \ \forall j\}$. Denote $\phi_L(x) = \phi(Lx)$ for all

$L > 0$. Fix $\delta_0 \leq \min\{\epsilon, \tilde{\delta}_1\}$, $r = n^{\frac{1}{2}} r_1$ and L large such that $r/L \leq \delta_0$. Then we have,

$$\begin{aligned}
\Big| \int_{Q_k} f(y) d\mu(y) \Big|^p &= 2^{p(n+1)} \Big| \int_{Q_k} \int_{A_{r_1}(0)} \phi(x) dx f(y) d\mu(y) \Big|^p \\
&= 2^{p(n+1)} L^{np} \Big| \int_{Q_k} \int_{A_{r_1/L}(y)} \phi_L(x-y) dx f(y) d\mu(y) \Big|^p \\
&= 2^{p(n+1)} L^{np} \Big| \int_{Q_k E_L} \int_{Q_k} \phi_L(x-y) f(y) d\mu(y) dx \Big|^p.
\end{aligned}$$

where $Q_k E_L = \{x = (x_1, ...x_n) \in M : \exists y = (y_1, ...y_n) \in E,$ such that $y_j - r_1/L < x_j \leq y_j \, \forall \, j\}$. Note that $|Q_k E_L| \leq |(Q_k \cap E)(r/L)|$, where $(Q_k \cap E)(r/L)$ denotes the r/L-distance set of $Q_k \cap E$ (since $r = n^{1/2} r_1$). Since ϕ and f are positive, $\int_{Q_k E_L} \int_{Q_k} \phi_L(x-y) f(y) d\mu(y) dx \leq \int_{Q_k E_L} \phi_L * f d\mu(x) dx$. Thus

$$\begin{aligned}
&\Big| \int_{Q_k} f(y) d\mu(y) \Big|^p \\
&\leq 2^{p(n+1)} L^{np} \Big| \int_{Q_k E_L} \phi_L * f d\mu(x) dx \Big|^p \\
&\leq 2^{p(n+1)} L^{np} (|Q_k E_L|)^{p-1} \int_{Q_k E_L} |\phi_L * f d\mu(x)|^p dx \\
&\leq 2^{p(n+1)} r^{(n-\alpha)(p-1)} L^{n+\alpha(p-1)} (|(Q_k \cap E)(r/L)| (r/L)^{(\alpha-n)(p-1)} \int_{Q_k E_L} |\phi_L * f d\mu(x)|^p dx.
\end{aligned}$$

By (3.2.0.9), there exists a constant $\tilde{C}$ independent of f, ϵ, and L such that

$$\frac{1}{\mu(Q_k)^{p-1}} \Big| \int_{Q_k} f(y) d\mu(y) \Big|^p \leq \tilde{C} L^{n+\alpha(p-1)} \int_{Q_k E_L} |\phi_L * f d\mu(x)|^p dx. \tag{3.2.0.12}$$

Let

$$e_{\epsilon_1} = \sum_{Q_k \in \mathcal{Q}_0} e_k = \sum_{Q_k \in \mathcal{Q}_0} \int_{Q_k} \Big| f(x) - \frac{1}{(\mu(Q_k))^p} \int_{Q_k} f(y) d\mu(y) \Big|^p d\mu(x). \tag{3.2.0.13}$$

Then from (3.2.0.8), (3.2.0.10) and (3.2.0.12), there exists a constant $\tilde{C}_1$ independent of f, ϵ and L such that

$$\begin{aligned}
&\int_E \frac{|f(x)|^p}{(\mu(E'_x) + \epsilon)^{2-p}} d\mu(x) \\
&\leq C_p \epsilon^{p-2} e_{\epsilon_1} + C_p \tilde{C} L^{n+\alpha(p-1)} \sum_{Q_k \in \mathcal{Q}_0} \int_{Q_k E_L} \frac{|\phi_L * f d\mu(x)|^p}{(\mu(E'_k) + \epsilon)^{2-p}} dx \\
&\leq C_p \epsilon^{p-2} e_{\epsilon_1} + \tilde{C}_1 L^{n(p-1)+\alpha} \sum_{Q_k \in \mathcal{Q}_0} \int_{Q_k E_L} \frac{|\phi_L * f d\mu(x)|^p}{(|E'_k(r/L)|)^{2-p}} dx, \qquad (3.2.0.14)
\end{aligned}$$

For given ϵ_1, let $g \in C_c^\infty(d\mu)$ be such that $\|f-g\|^p_{L^p(d\mu)} < \epsilon_1$. Then, as in the proof of Theorem 3.1.6, we have

$$\begin{aligned} e_{\epsilon_1} &\leq 2C_p^2 \sum_k \int_{Q_k} |f(x)-g(x)|^p d\mu(x) \\ &\quad + C_p \sum_k \int_{Q_k} \left| g(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} g(y) d\mu(y) \right|^p d\mu(x). \end{aligned}$$

Since $E = \cup_k (Q_k \cap E)$ and $\mu = \mathcal{P}^\alpha|_E$,

$$\begin{aligned} e_{\epsilon_1} &\leq 2C_p^2 \|f-g\|^p_{L^p(d\mu)} + C_p \sum_k \int_{Q_k} \left| g(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} g(y) d\mu(y) \right|^p d\mu(x) \\ &\leq 2C_p\epsilon_1 + 2\sum_k \int_{Q_k} \left| g(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} g(y) d\mu(y) \right|^2 d\mu(x). \end{aligned} \quad (3.2.0.15)$$

Since g is compactly supported continuous function, g is uniformly continuous and

$$|g(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} g(y) d\mu(y)| \to 0$$

uniformly in x and Q_k as $\mu(Q_k) \to 0$. As $\epsilon_1 \to 0$, we have $\mu(Q_k) \to 0$ for all k. Hence

$$\begin{aligned} & \sum_k \int_{Q_k} \left| g(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} g(y) d\mu(y) \right|^p d\mu(x) \\ \leq\ & \sum_k \mu(Q_k) \sup_{Q_k \in \mathcal{Q}_0} \sup_{x \in Q_k} \left| g(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} g(y) d\mu(y) \right|^p \\ =\ & \mu(E) \sup_{Q_k \in \mathcal{Q}_0} \sup_{x \in Q_k} \left| g(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} g(y) d\mu(y) \right|^p, \end{aligned}$$

which goes to zero as ϵ_1 goes to zero. Therefore, from (3.2.0.15), e_{ϵ_1} goes to zero as ϵ_1 goes to zero.

Since $r/L < \epsilon$, for each $k = (k_1, ...k_n)$, $Q_k E_L$ intersects with at most $2^n - 1$ other cubes $Q_m \cap E(r/L)$, where $m = (m_1, ..., m_n)$, $k_j - 1 \leq m_j \leq k_j$. Hence for each k, $Q_k E_L$ is the union of $Q_k \cap E(r/L)$ and at most $2^n - 1$ other sets $Q_m \cap E(r/L)$. Then for all such m, $|E'_m(r/L)| \leq |E'_k(r/L)|$. Thus for each k,

$$\int_{Q_k \cap E(r/L)} \frac{|\phi_L * f d\mu(x)|^p}{|E'_k(r/L)|^{2-p}} dx$$

repeats at most 2^n times. Let $\tilde{\mathcal{Q}}_0$ denote the collection of all $Q_k = \{x = (x_1, ...x_n) \in M : (k_j - 1)\epsilon_1 < x_j \leq k_j \epsilon_1\}$ where $k = (k_1, ...k_n)$, $(0 < k_j \in \mathbb{Z})$ such that $|Q_k \cap E(r/L)| \neq 0$. Thus

from (3.2.0.14) and (3.2.0.8), there exists a constant C_0 independent of f, ϵ and L such that for all $r/L \leq \delta_0$,

$$\begin{aligned}&\int_E \frac{|f(x)|^p}{(\mu(E'_x)+\epsilon)^{2-p}} d\mu(x)\\ &\leq C_p \epsilon^{p-2} e_{\epsilon_1} + C_0 L^{n(p-1)+\alpha} \sum_{Q_k \in \tilde{\mathcal{Q}}_0} \int_{Q_k \cap E(r/L)} \frac{|\phi_L * f d\mu(x)|^p}{|E'_k(r/L)|^{2-p}} dx, \qquad (3.2.0.16)\end{aligned}$$

where e_{ϵ_1} goes to zero as ϵ_1 goes to zero.

Denote $\delta = r/L$. By the construction of C_ϵ, for all k, $|C_\epsilon(\delta)| < |E'_k(\delta)|$. Also, by Lemma 1.1.3, $|C_\epsilon(\delta)| \geq C_n P(C_\epsilon, \delta)\delta^n$. Denote $P_\delta = P(C_\epsilon, \delta) > 1$, the δ-packing number of C_ϵ. For $j = 0, 1, ...J$, let $\mathcal{S}_j$ be the sub-collection of all $Q_k \in \tilde{\mathcal{Q}}_0$ such that $2^j P_\delta \delta^n \leq |E'_k(\delta)| < 2^{j+1} P_\delta \delta^n$. We consider only nonempty collections. Denote $g_L(x) = \phi_L * f d\mu(x)$. Then

$$\begin{aligned}\sum_{Q_k \in \tilde{\mathcal{Q}}_0} \int_{Q_k \cap E(\delta)} \frac{|g_L(x)|^p}{|E'_k(\delta)|^{2-p}} &= \sum_j \sum_{Q_k \in \mathcal{S}_j} \int_{Q_k \cap E(\delta)} \frac{|g_L(x)|^p}{|E'_k(\delta)|^{2-p}}\\ &\leq \sum_j (2^j P_\delta \delta^n)^{p-2} \sum_{Q_k \in \mathcal{S}_j} \int_{Q_k \cap E(\delta)} |g_L(x)|^p dx.\end{aligned}$$

For each j, we can write $\cup_{Q_k \in \mathcal{S}_j} Q_k \cap E(\delta) = S_j = \cup_{i=1}^{s_j} \Delta_i^\delta$ as the finite disjoint union of non-empty sets intersected with cubes of volume δ^n, that is, $0 < |\Delta_i^\delta| \leq \delta^n$. Then

$$\sum_{Q_k \in C\tilde{Q}_0} \int_{Q_k \cap E(\delta)} \frac{|g_L(x)|^p}{|E'_k(\delta)|^{2-p}} \leq \sum_j (2^j P_\delta \delta^n)^{p-2} \int_{S_j} |g_L(x)|^p dx \qquad (3.2.0.17)$$

For every j, applying Lemma 3.2.9, we have

$$\frac{\delta^{-n}}{P_\delta} \int_{S_j} |g_L(x)| dx \leq C \int_{\mathbb{R}^n} |\widehat{g_L}(\xi)| d\xi. \qquad (3.2.0.18)$$

We recall the following interpolation theorem due to Stein (See page 213 in [4] for the proof):

Theorem 3.2.10. *Let $(\mathcal{R}, \mu)$ and $(\mathcal{S}, \nu)$ be totally σ-finite measure spaces and let T be a linear operator defined on the μ-simple functions on $\mathcal{R}$ taking values in the ν-measurable functions on $\mathcal{S}$. Suppose that u_i, v_i are positive weights on $\mathcal{R}$ and $\mathcal{S}$ respectively, and that $1 \leq p_i, q_i \leq \infty$, $(i = 0, 1)$. Suppose*

$$\|(Tf)v_i\|_{q_i} \leq M_i \|f u_i\|_{p_i}, \ \ (i = 0, 1)$$

for all μ-simple functions f. Let $0 \le \theta \le 1$ and define

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

and

$$u = u_0^{1-\theta} u_1^{\theta}, \quad v = v_0^{1-\theta} v_1^{\theta}.$$

Then, if $p < \infty$, the operator T has a unique extension to a bound linear operator from L_u^p into L_v^q which satisfies

$$\|(Tf)v\|_q \le \ M_0^{1-\theta} M_1^{\theta} \|fu\|_p,$$

for all $f \in L_u^p$

Let $v_0 = \frac{\delta^{-n}}{P_\delta} \chi_{S_j}(x)$ and $v_1 = u_0 = u_1 = 1$, where χ_{S_j} denotes the characteristic function on S_j. Let T be defined as $T(\psi) = \check{\psi}$, the inverse Fourier transform of ψ. By (3.2.0.18), we have for each j and L,

$$\|(Tg_L)v_0\|_1 \le C \|\widehat{g_L}\|_1$$

By Plancherel theorem, we have

$$\|(Tg_L)v_1\|_2 \le \|\widehat{g_L}\|_2$$

Then applying the Theorem 3.2.10, for $1 < p < 2$, we have

$$(\delta^n P_\delta)^{p-2} \int_{S_j} |\phi_L * f d\mu(x)|^p dx \le \ C' \int_{\mathbb{R}^n} |\widehat{\phi_L * f d}\mu(\xi)|^p d\xi. \tag{3.2.0.19}$$

where C' is a non-zero finite constant independent of f. Using (3.2.0.17), (3.2.0.16) and (3.2.0.19), there exists a constant C independent of f, ϵ and L such that for very large L

$$\begin{aligned} &\int_E \frac{|f(x)|^p}{(\mu(E'_x) + 2\epsilon)^{2-p}} d\mu(x) \\ &\le \ e_{\epsilon_1} \epsilon^{p-2} + C L^{n(p-1)+\alpha} \int_{\mathbb{R}^n} |\widehat{\phi_L * f d}\mu(\xi)|^p d\xi. \end{aligned}$$

Since ϕ is a Schwartz function such that $\widehat{\phi}$ is supported in unit ball, $\|\widehat{\phi}\|_\infty \le 1$ and $\widehat{\phi_L}(\xi) = L^{-n} \widehat{\phi}(L^{-1}\xi)$,

$$\int_E \frac{|f(x)|^p}{(\mu(E'_x) + 2\epsilon)^{2-p}} d\mu(x) \ \le \ e_{\epsilon_1} \epsilon^{p-2} + C \ L^{\alpha + n(p-1)} \int_{|\xi| \le L} \frac{|\widehat{f d\mu}(\xi)|^p}{L^{np}} d\xi,$$

for all $r/L \le \delta_0$, where δ_0 goes to zero as $\epsilon_1 < \epsilon/2 \to 0$. Hence letting ϵ_1 to zero, we have

$$\int_E \frac{|f(x)|^p}{[\mu(E'_x) + 2\epsilon]^{2-p}} d\mu(x) \le \ C \liminf_{L \to \infty} \frac{1}{L^{n-\alpha}} \int_{|\xi| \le L} |\widehat{f d\mu}(\xi)|^p d\xi.$$

Letting ϵ go to zero, using (3.2.0.7), we have

$$\int_E \frac{|f(x)|^p}{[\mu(E_x)]^{2-p}} d\mu(x) \leq \; C \liminf_{L\to\infty} \frac{1}{L^{n-\alpha}} \int_{|\xi|\leq L} |\widehat{fd\mu}(\xi)|^p d\xi.$$

Hence the proof.

Theorem 3.2.11. *Let $E \subset \mathbb{R}^n$ be a compact quasi α-regular set of non-zero finite α-dimensional Hausdorff measure and $\mu = \mathcal{H}_\alpha|_E$. Let $f \in L^p(d\mu)$ for $1 \leq p \leq 2$. Then there exists a constant C independent of f such that*

$$\int_E \frac{|f(x)|^p}{[\mu(E_x)]^{2-p}} d\mu(x) \leq \; C \liminf_{L\to\infty} \frac{1}{L^{n-\alpha}} \int_{|\xi|\leq L} |\widehat{fd\mu}(\xi)|^p d\xi, \tag{3.2.0.20}$$

where $E_x = E \cap [(-\infty, x_1] \times ... \times (-\infty, x_n]]$ for all $x = (x_1, ...x_n) \in \mathbb{R}^n$.

Proof. The proof follows as in the Theorem 3.2.8. The hypothesis E has finite α-packing measure was used only when we invoked Lemma 2.1.1. In the present case, we can use Lemma 2.1.3. $\square$

Chapter 4

Applications to Wiener Tauberian type theorems

A classical result of Wiener[44] states that the translates of a function $f \in L^1(\mathbb{R}^n)$ spans a dense subset of $L^1(\mathbb{R}^n)$ if and only if the Fourier transform of f, $\widehat{f}(t) \neq 0 \; \forall \; t \in \mathbb{R}^n$. In fact, if ${}^x f(y) = f(y-x)$ and $g \in L^\infty(\mathbb{R}^n)$ is such that $\int_{\mathbb{R}^n} {}^x f(y) g(y) dy = 0 \, \forall \, x \in \mathbb{R}^n$, we get $\widetilde{f} * g = 0$ where $\widetilde{f}(t) = f(-t)$. Distribution theory tells us that $supp \; \widehat{g} \subseteq \{x \in \mathbb{R}^n : \widehat{f}(x) = 0\}$ (which is Wiener Tauberian theorem in disguise. See [33]). If $\widehat{f}$ is nowhere vanishing then it follows that $g \equiv 0$. This crucial step in the proof of Wiener's theorem leads us to the study of functions f in $L^p(\mathbb{R}^n)$ with $supp \; \widehat{f}$ in a thin set. Thus Theorem 2.2.4 can be used to prove Wiener-Tauberian type theorems. In this chapter, we apply our results to prove Wiener Tauberian type theorems on $\mathbb{R}^n$ and $M(2)$.

4.1 L^p Wiener Tauberian Theorems on $\mathbb{R}^n$

In this section, we improve the results on L^p versions of Wiener Tauberian type theorems on $\mathbb{R}^n$ obtained in [31]. Consider the motion group $M(n) = \mathbb{R}^n \rtimes SO(n)$ with the group law

$$(x_1, k_1)(x_2, k_2) = (x_1 + k_1 x_2, k_1 k_2).$$

For a function h on $\mathbb{R}^n$ and an arbitrary $g = (y, k) \in M(n)$, let ${}^g h$ be the function ${}^g h(x) = h(kx + y), \; x \in \mathbb{R}^n$. Let $\widehat{h}$ denote the Euclidean Fourier transform of the function h. For $h \in$

$L^1 \cap L^p(\mathbb{R}^n),\ 1 \leq p \leq \infty$, let $S = \{r > 0 : \widehat{h} \equiv 0 \text{ on } C_r\}$, where C_r is the sphere of radius $r > 0$ centered at origin in $\mathbb{R}^n$. Let $Y = Span\{{}^g h : g \in M(n)\}$. Then the main result from [31] is

Theorem 4.1.1. *1. If $p = 1$, then Y is dense in $L^1(\mathbb{R}^n)$ if and only if S is empty and $\widehat{h}(0) \neq 0$.*

2. *If $1 < p < \frac{2n}{n+1}$, then Y is dense in $L^p(\mathbb{R}^n)$ if and only if S is empty.*

3. *If $\frac{2n}{n+1} \leq p < 2$, and every point of S is an isolated point, then Y is dense in $L^p(\mathbb{R}^n)$.*

4. *If $2 \leq p \leq \frac{2n}{n-1}$, and S is of zero measure in $\mathbb{R}^+$, then Y is dense in $L^p(\mathbb{R}^n)$.*

5. *If $\frac{2n}{n-1} < p < \infty$, then Y is dense in $L^p(\mathbb{R}^n)$ if and only if S is nowhere dense.*

We show that the part (3) of the above theorem can be improved:

Theorem 4.1.2. *[37] Let $f \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ and let $S = \{r > 0 : \widehat{f} \equiv 0 \text{ on } C_r\}$ be such that $\mathcal{P}^\beta(S) < \infty$, for some $0 \leq \beta < 1$. If $\frac{2n}{n+1-\beta} \leq p \leq 2$, then $Y = Span\{{}^g f : g \in M(n)\}$ is dense in $L^p(\mathbb{R}^n)$.*

Proof. Fix $\epsilon < 1$. Suppose Y is not dense in $L^p(\mathbb{R}^n)$. Let $h \in L^q(\mathbb{R}^n)$ annihilate all the elements in Y, where $\frac{1}{p} + \frac{1}{q} = 1$. We can assume h to be smooth, bounded and radial (See the arguments in [31]). It follows that $h * f \equiv 0$. Then $supp\, \widehat{h}$ is contained in the zero set of $\widehat{f}$. Let α and q be such that $2 \leq q = \frac{2n}{\alpha} \leq \frac{2n}{n-1+\beta}$. Choose an even function $\chi \in C_c^\infty(\mathbb{R}^n)$ with support in the unit ball and $\int_{\mathbb{R}^n} \chi(x)dx = 1$. Let $\chi_\epsilon(x) = \epsilon^{-n}\chi(x/\epsilon)$ and $u_\epsilon = u * \chi_\epsilon$ where $u = \widehat{h}$. Since $2 \leq q$, as in Lemma 2.2.3,

$$\begin{aligned} \|u_\epsilon\|^2 &\leqslant C\epsilon^{\alpha-n} \sum_{j=-\infty}^{\infty} a_j b_j^\epsilon, \\ \text{where } a_j &= 2^{j(n-\alpha)} \sup_{2^j \leqslant |x| \leqslant 2^{j+1}} |\widehat{\chi}(x)|^2 \\ \text{and } b_j^\epsilon &= (2^{-j}\epsilon)^{n-\alpha} \int_{2^j \leqslant |\epsilon x| \leqslant 2^{j+1}} |h(x)|^2 dx. \end{aligned}$$

and $\sum\limits_{j=-\infty}^{\infty} a_j b_j^\epsilon \to 0$ as $\epsilon \to 0$.

Let $\psi \in C_c^\infty(\mathbb{R}^n)$. Let $M = supp\, \widehat{h} \cap supp\, \psi$ and let $R_\psi > 0$ be such that M is contained in a ball of radius R_ψ. For $x \in M$, $\|x\| \in S$ and $\|x\| \leq R_\psi$. Let $S_\psi = \{r \in S : r \leq R_\psi\}$. Then

S_ψ is a bounded subset of S. We claim that

$$\lim_{\epsilon\to 0}\epsilon^{\beta-1}\int_{M_\epsilon}|\psi(x)|^2dx<\infty. \tag{4.1.0.1}$$

Then,

$$\begin{aligned}
|<u,\psi>|^2 &= \lim_{\epsilon\to 0}|<u_\epsilon,\psi>|^2\\
&\le \lim_{\epsilon\to 0}\|u_\epsilon\|_2^2\int_{M_\epsilon}|\psi|^2\\
&\le C\lim_{\epsilon\to 0}\epsilon^{\alpha-n}\sum_{j=-\infty}^{\infty}a_jb_j^\epsilon\int_{M_\epsilon}|\psi(x)|^2dx\\
&\le C\lim_{\epsilon\to 0}\epsilon^{\alpha-n-\beta+1}\epsilon^{\beta-1}\sum_{j=-\infty}^{\infty}a_jb_j^\epsilon\int_{M_\epsilon}|\psi(x)|^2dx,
\end{aligned}$$

since $2\le\frac{2n}{\alpha}\le\frac{2n}{n-1+\beta}$, that is $0\le\alpha-n-\beta+1$ and $\lim\limits_{\epsilon\to 0}\sum_j a_jb_j^\epsilon=0$ we get $u\equiv 0$ and hence $h\equiv 0$.

Proof of (4.1.0.1): The proof is similar to that of Lemma 2.1.1. Since $\mathcal{P}^\beta(S_\psi)\le\mathcal{P}^\beta(S)<\infty$, let $\{A_i\}$ be a cover of S_ψ such that $\sum_i P_0^\alpha(A_i)<\infty$. Then $P_0^\alpha(A_i\cap S_\psi)<\infty$. For $S_\psi^i=A_i\cap S_\psi$, let $P(S_\psi^i,\epsilon)$ be the maximum number of disjoint balls with centers $\{r_j\}$ in S_ψ^i, of radius ϵ and $N(S_\psi^i,\epsilon)$ be the ϵ-covering number of S_ϵ^i. Then

$$\begin{aligned}
S_\psi^i &\subseteq \cup_{j=1}^{N(S_\psi^i,\epsilon)}(r_j-\epsilon/2,r_j+\epsilon/2)\quad\text{and}\\
S_\psi(\epsilon) &\subset \cup_i S_\psi^i(\epsilon)\subseteq\cup_i\cup_{j=1}^{N(S_\psi^i,\epsilon)}(r_j-\epsilon,r_j+\epsilon).
\end{aligned}$$

If $x\in M(\epsilon)$, then $\|x\|\in S_\psi(\epsilon)$. We have,

$$\begin{aligned}
\int_{M_\epsilon}|\psi(x)|^2dx &\le \int_{r\in S_\psi(\epsilon)}\int|\psi(r\omega)|^2d\omega r^{n-1}dr\\
&\le (R_\psi+\epsilon)^{n-1}\int_{r\in S_\psi(\epsilon)}\int|\psi(r\omega)|^2d\omega dr\\
&\le (R_\psi+1)^{n-1}\|\psi\|_\infty^2\Omega_n\sum_i\sum_{j=1}^{N(S_\psi^i,\epsilon)}\int_{r_j-\epsilon}^{r_j+\epsilon}dr\\
&= C_1\sum_i N(S_\psi^i,\epsilon)(2\epsilon)\\
&\le 2C_1\epsilon\sum_i P(S_\psi^i,\epsilon/2)\quad\text{(by Lemma 1.1.3)}
\end{aligned}$$

where $C_1 = (R_\psi + 1)^{n-1} \|\psi\|_\infty^2 \Omega_n$ is a constant independent of ϵ and Ω_n is the volume of the unit sphere in $\mathbb{R}^n$. $\square$

Remark 4.1.3. *Suppose S is isolated. Convolving f with an arbitrary Schwartz class function whose Fourier transform is compactly supported, we may reduce to the case where S is finite. The case $\beta = 0$ in the above theorem then implies part (3) of Theorem 4.1.1.*

Now let f be a function in $L^1 \cap L^p(\mathbb{R})$ and let F denote the closed set where the Fourier transform of f vanishes. In [6], A. Beurling proved that if for some p in $(1, 2)$, the space of finite linear combinations of translates of f is not dense in $L^p(\mathbb{R})$, then the Hausdorff dimension of F is at least $2 - (2/p)$ (See also page 312 in [8]). In other words, if the Hausdorff dimension of F is α, for $0 \le \alpha \le 1$, then the space of finite linear combinations of translates of f is dense in $L^p(\mathbb{R})$ for $2/(2-\alpha) < p < \infty$. Now using Theorem 2.2.4, we prove a similar result (including the end points for the range) on $\mathbb{R}^n$ where Hausdorff dimension is replaced with the packing dimension.

Theorem 4.1.4. *Let $f \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ for $\frac{2n}{2n-\alpha} \le p < \infty$ and let the zero set of $\widehat{f} \subseteq E$, where $\mathcal{P}^\alpha(E) < \infty$ for some $0 \le \alpha < n$. Then $X = span\{{}^x f : x \in \mathbb{R}^n\}$ is dense in $L^p(\mathbb{R}^n)$.*

Proof. Suppose X is not dense in $L^p(\mathbb{R}^n)$. Then as above, there exists a non trivial, smooth and radial $h \in L^q(\mathbb{R}^n)$ such that $h * f_1 \equiv 0$ for all $f_1 \in X$. Clearly the zero set of $X (\subset L^1(\mathbb{R}^n))$, $\bigcap_{u \in X} \{s \in \mathbb{R}^n : \widehat{u}(s) = 0\}$ is equal to the zero set of $\widehat{f}$, $Z(\widehat{f})$. Hence $supp\ \widehat{h} \subseteq Z(\widehat{f})$. Since $\frac{2n}{2n-\alpha} \le p < \infty$, we have $1 < q \le \frac{2n}{\alpha}$. By Theorem 2.2.4, $h = 0$. Thus X is dense in $L^p(\mathbb{R}^n)$. $\square$

In [15], C. S Herz studied versions of L^p- Wiener Tauberian theorems. From Theorem 1 and Theorem 4 of [15], we note that for $f \in L^1 \cap L^p(\mathbb{R}^n),\ p < \infty$ the alternative sufficient conditions for the translates of f to span L^p are,

1. $|K(\epsilon)| = o(\epsilon^{n(1-2/q)})$ for each compact subset K of E.
2. $\dim_H(E) = \alpha < 2n/q$, with the proviso, if $n > 2$, that $q \le 2n/(n-2)$.

where E denotes the zero set of $\widehat{f}$ and $\frac{1}{p} + \frac{1}{q} = 1$. With an additional hypothesis on E, using Theorem 4.1.4, we can improve the result in [15]:

Proposition 4.1.5. *For $f \in L^1 \cap L^p(\mathbb{R}^n),\ 1 \le p < \infty$ a sufficient condition that the translates of f span L^p is : the zero set of $\widehat{f}$ has finite packing α- measure for $\alpha \le 2n/q$ where $\frac{1}{p} + \frac{1}{q} = 1$.*

4.2 L^p Wiener Tauberian Theorem on $M(2)$

In this section, we look at one sided and two sided analogues of Wiener Tauberian Theorems on $M(2)$ and improve a few results from [28].

The group $M(2)$ is the semi-direct product of $\mathbb{R}^2$ with the special orthogonal group $K = SO(2)$. The group law in $G = M(2)$ is given by

$$(z, e^{i\alpha})(w, e^{i\beta}) = (z + e^{i\alpha}w, e^{i(\alpha+\beta)}).$$

The Haar measure on G is given by $dg = dzd\alpha$ where dz is the Lebesgue measure on $\mathbb{C}$ and $d\alpha$ is the normalized Haar measure on S^1. For each $\lambda > 0$, we have a unitary irreducible representation of G realized on $H = L^2(K) = L^2([0, 2\pi], dt)$, given by

$$[\pi_\lambda(z, e^{it})u](s) = e^{i\lambda < z, e^{is} >}u(s - t),$$

for $(z, e^{it}) \in G$ and $u \in H$. Here $< z, w >=$Re$(z.\bar{w})$. It is known that these are all the infinite dimensional, non equivalent unitary irreducible representations of G. Apart from the above family, we have another family $\{\chi_n, n \in \mathbb{Z}\}$ ($\mathbb{Z}$ is the set of integers) of one dimensional unitary representations of G, given by $\chi_n(z, e^{i\alpha}) = e^{in\alpha}$. Then the unitary dual $\widehat{G}$, of G is the collection $\{\pi_\lambda, \lambda > 0\} \cup \{\chi_n : n \in \mathbb{Z}\}$ (See page 165, [42]). For $f \in L^1(G)$, define the **"group theoretic" Fourier transform** of f as follows:

$$\pi_\lambda(f) = \int_G f(g)\pi_\lambda(g)dg, \ \lambda > 0 \text{ and}$$

$$\chi_n(f) = \int_G f(z, e^{i\alpha})e^{-in\alpha}dzd\alpha, \ n \in \mathbb{Z}.$$

From the Plancherel theorem for G (see page 183, [42]) we have for $f \in L^2(G)$,

$$\|f\|_2^2 = \int_0^\infty \|\pi_\lambda(f)\|_{HS}^2 \lambda d\lambda,$$

where $\|.\|_{HS}$ denotes the Hilbert-Schmidt norm.

For $g_1, g_2 \in G$, the two sided translate, ${}^{g_1}f^{g_2}$ of f is the function defined by ${}^{g_1}f^{g_2}(g) = f(g_1^{-1}gg_2)$. For $f \in L^1(G) \cap L^p(G)$, let $S = \{a > 0 : \pi_a(f) = 0\}, \ X = Span\ \{{}^{g_1}f^{g_2} : g_1, g_2 \in G\}, \ S' = \{\lambda > 0 :$ Range of $\pi_\lambda(f)$ is not dense$\}$ and V_f be the closed subspace spanned by the right translates of f in $L^p(G)$.

Theorem 4.2.1. *Let $f \in L^1(G) \cap L^p(G)$.*

1. *For $\frac{4}{3-\alpha} \leq p < 2$, if $S = \{a > 0 : \pi_a(f) = 0\}$ is such that $\mathcal{P}^\alpha(S) < \infty$ for $0 \leq \alpha < 1$, then $X = span\{{}^{g_1}f^{g_2} : g_1, g_2 \in M(2)\}$ is dense in $L^p(M(2))$.*

2. *If f is radial in the $\mathbb{R}^2$ variable and $\mathcal{P}^\alpha(S') < \infty$ for some $0 \leq \alpha < 1$, then $V_f = L^p(M(2))$ provided $\frac{4}{3-\alpha} \leq p \leq 2$.*

Proof. To prove part (1), we proceed as in the proof of Theorem 2.1 in [28]. First we prove $L^p(G/K) \subseteq \overline{X}$.

For $f \in L^1(G)$, the operator $\pi_a(f)$ is well defined for each $a > 0$. Suppose $\pi_a(f) \neq 0$, then there exists $w \in H = L^2(K) = L^2([0, 2\pi], dt)$ such that $\pi_a(f)(w) \neq 0$.

For given $a, \epsilon > 0$, since π_a is irreducible, there exists constants $c_1, c_2, ..., c_m$ and elements x_1, x_2, ... $x_m \in G$, such that $\| \sum_{j=1}^m c_j \pi_a(x_j) v_0 - w\| < \epsilon$, where v_0 is K-fixed vector $v_o \equiv 1 \in H$. Therefore we have,

$$\Big||\pi_a(f) \sum_{j=1}^{m} c_j \pi_a(x_j) v_0 - \pi_a(f) w\Big|| < \epsilon ||\pi_a(f)||$$

Define $F_a = \sum_{j=1}^m c_j f^{x_j^{-1}}$. Then $||\pi_a(F_a)v_0 - \pi_a(f)w|| < \epsilon||\pi_a(f)||$ and $\pi_a(F_a)v_0 \neq 0$ for small enough ϵ. Let

$$F_a^{\#}(x) = \int_K F_a(xk)dk, \; x \in G.$$

Then $F_a^{\#}$ is a right K-invariant function on G. $\pi_a(F_a^{\#})v_0 = \pi_a(F_a)v_0 \neq 0$ and $[\pi_a(F_a^{\#})(v_0)](s) = \hat{F_a^{\#}}(ae^{is})$ implies $\hat{F_a^{\#}}$ is non identically zero on the sphere $\{x \in \mathbb{R}^2 : ||x|| = a\}$. Thus whenever $\pi_a(f) \neq 0$, we have a right K-invariant function $F_a^{\#}$ which can be considered as a function on $\mathbb{R}^2$, that is $F_a^{\#} \in L^1(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$ such that its Euclidean Fourier transform is not identically zero on the sphere $C_a = \{x \in \mathbb{R}^2 : \|x\| = a\}$.

Define $S_1 = \cap_{a \in S^c}\{r > 0 : \widehat{F_a^{\#}} \equiv 0 \text{ on } C_r\}$. Then $S_1 \subset S$. We have

$$\overline{Span\{{}^{g}F_a^{\#} : g \in G, a \in S_1^c\}} \subseteq \overline{Span\{{}^{g_1}f^{g_2} : g_1, g_2 \in G\}}.$$

Also using Theorem 4.1.2,

$$\overline{Span\{{}^gF_a^{\#} : g \in G, a \in S_1^c\}} = L^p(G/K).$$

Thus $L^p(G/K) \subseteq \overline{Span\{{}^{g_1}f^{g_2} : g_1, g_2 \in G\}} = \overline{X}$.

Suppose $\overline{X} \neq L^p(G)$, then let $\Phi \in L^q(G)$ be such that

$$\int_G \psi(g)\Phi(g)dg = 0 \; \forall \psi \in \overline{X},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Convolving Φ with an approximate identity we can assume $\Phi \in L^q \cap L^\infty(G)$. Since $\overline{X}$ is invariant under right translations by G, there exists an integer m and a non trivial $\phi \in L^q \cap L^\infty(\mathbb{R}^2)$ such that

$$\Phi(g) = \Phi(z, e^{is}) = \phi(z)e^{ims}$$

and

$$\int_G \psi(z, e^{is})\phi(z)e^{ims}dzds = 0 \; \forall \psi \in \overline{X}.$$

Since $L^p(G/K) \subseteq \overline{X}$, choose a rapidly decaying function on $\mathbb{R}^2$ of the form $h(z) = h(re^{i\theta}) = h_1(r)e^{in\theta}$. Then for $\psi_w(z, e^{is}) = h(z + e^{is}w) \in \overline{X}$ for all $w \in \mathbb{R}^2$ and hence

$$\int_G h(z + e^{is}w)\phi(z)e^{ims}dzds = 0.$$

Since $h(e^{is}z) = e^{ins}h(z)$, we deduce from the above equality that $h *_{\mathbb{R}^2} \phi_n \equiv 0$ for $\phi_n \in L^q \cap L^\infty(\mathbb{R}^2)$ such that

$$\phi_n(z) = \int_0^{2\pi} \phi(e^{is}z)e^{i(m+n)s}ds.$$

Since $\psi_w(z, e^{is}) = h(z + e^{is}w) \in \overline{X}$ and h is radial, the zeroes of $\hat{h}$ is contained in $\{x \in \mathbb{R}^2 : ||x|| \in S\}$. Hence $supp\ \hat{\phi_n}$ is contained in $\{x \in \mathbb{R}^2 : ||x|| \in S\}$. By the assumptions on p and q, using Theorem 4.1.2 $\phi_n \equiv 0$ for all n. This contradicts the assumption that ϕ is non trivial. Hence $\overline{X} = L^p(G)$.

To prove part(2), we proceed as in the proof of (c) of Theorem 3.2 in [28]. Let $\phi(z)e^{im_0\alpha} \in L^q \cap L^\infty(M(2))$ kill all the functions in V_f where $\frac{1}{p} + \frac{1}{q} = 1$. Then f being radial in the $\mathbb{R}^2$-variable we are led to the convolution equation $f_m *_{\mathbb{R}^2} \phi_m = 0$ where ϕ_m is defined by

$$\phi_m(z) = \int_0^{2\pi} \phi(e^{i\alpha}z)e^{i(m_0+m)\alpha}d\alpha.$$

and f_m is defined by

$$f_m(z) = \int_{S^1} f(z, e^{i\alpha}) e^{-im\alpha} d\alpha.$$

Taking Fourier transform we obtain that $supp\ \widehat{\phi}_m$ is contained in $\{z \in \mathbb{R}^2 : \|z\| \in S\}$. Proceeding as in the proof of Theorem 4.1.2, we have $< \phi_m, \psi >= 0$ for all $\psi \in C_c^\infty(\mathbb{R}^2)$ and m. Thus $\phi_m \equiv 0$ for all m.

$\square$

Further Questions

In this chapter, we will briefly describe some problems which are related to the results discussed in this thesis.

(I) Recall that for a positive Radon measure μ with compact support $E \subset \mathbb{R}^n$, the α-energy, $I_\alpha(\mu)$ is given by

$$\begin{aligned} I_\alpha(\mu) &= \int_E \int_E |x-y|^{-\alpha} d\mu(x) d\mu(y) \\ &= C \int_{\mathbb{R}^n} |\xi|^{\alpha-n} |\widehat{\mu}(\xi)|^2 d\xi, \end{aligned}$$

where the constant C depends only on n and α. Let $\sigma_\mu(r) = \int_{S^{n-1}} |\widehat{\mu}(r\omega)|^2 d\omega$. Thus if $I_\alpha(\mu) < \infty$, there exists constant C_μ depending only on μ such that,

$$\begin{aligned} |\widehat{\mu}(x)|^2 &\leq C_\mu |x|^{-\alpha}, \\ \sigma_\mu(r) &\leq C_\mu r^{-\alpha}, \end{aligned}$$

for most x and r (See Chapter 12 in [24]). In [25], the author proved that

$$\sigma_\mu(r) \leq C I_\alpha(\mu) r^{-\alpha} \text{ for all } 0 < r < \infty,\ 0 < \alpha \leq \frac{1}{2}(n-1).$$

See also [38], [39], [10], [11] and [46] for similar results.

If the restriction exponent $p(n,\alpha,\beta)$ is defined by

$$\begin{aligned} p(n,\alpha,\beta) &= \inf\{q : (\forall\mu \text{ with } \mu(B_r(x)) \leq r^\alpha \text{ and } |\widehat{\mu}(\xi)|^2 \leq |\xi|^{-\beta}) \\ &\quad (\forall f \in L^2(d\mu))(\|\widehat{fd\mu}\|_q \leq C_{q,\mu}\|f\|_{L^2(d\mu)})\}, \end{aligned}$$

for $0 < \alpha, \beta < n$, then Mitsis proved in the Proposition 3.1 in [26] that $p(n,\alpha,\beta) \geq \frac{2n}{\alpha}$. See also in [27] and [3].

Theorem[26]: *Let μ be a measure in $\mathbb{R}^n$ such that*

$$\begin{aligned} \mu(B_r(x)) &\leq C_1 r^\alpha \; \forall\, x \in \mathbb{R}^n \text{ and } r > 0, \\ |\widehat{\mu}(\xi)| &\leq |\xi|^{-\beta/2}, \; \forall\, \xi \in \mathbb{R}^n \end{aligned}$$

for some $0 < \alpha < n$. Then for every $p \geq \frac{2(2n-2\alpha+\beta)}{\beta}$, there exists a constant $C_p > 0$ such that

$$\|\widehat{fd\mu}\|_p \leq C_{p,n,\alpha}\|f\|_{L^2(d\mu)}, \tag{5.1}$$

for all $f \in L^2(d\mu)$.

When $\beta = \alpha$, we have the result for $p \geq \frac{2(2n-\alpha)}{\alpha}$. In [14], the authors proved the sharpness of the above theorem for $n = 1$ by constructing a probability measure μ on a set of dimension α that satisfies the hypothesis of the above theorem but fails (5.1) for $p < \frac{2(2-\alpha)}{\alpha}$. Note that $\frac{2n}{\alpha} < \frac{2(2n-\alpha)}{\alpha}$. In this thesis, we looked at only the range $p < \frac{2n}{\alpha}$ and obtained bounds for

$$\frac{1}{L^k}\int_{|\xi|\leq L} |\widehat{fd\mathcal{P}^\alpha|_E}(\xi)|^p d\xi,$$

where E is a compact set of finite α-packing measure. We would like to analyze the behaviour of L^p-average of $\widehat{fd\mu}$ over a ball of large radius for $p < \frac{2(2n-\alpha)}{\alpha}$, if E is a set of finite α-packing measure and μ is a measure supported on E such that $|\widehat{\mu}(\xi)| \leq |\xi|^{-\alpha/2}$.

(II) In Lemma 2.1.1 we proved that if E is a set of finite α-packing measure, then $|S(\delta)|\delta^{\alpha-n}$ is bounded above by the packing measure of S for all bounded subsets S of E as δ approaches zero. In [17], the authors called a set $E \subset \mathbb{R}$ of finite α-dimensional Hausdorff measure, an α-coherent set, if for every $x \in \mathbb{R}$, $|E_x(\delta)|\delta^{\alpha-1}$ is bounded above by the Hausdorff measure of E_x as

δ approaches zero where $E_x = E \cap (-\infty, x]$. The author in [45], introduced curvature measures for the fractals. k^{th} average fractal curvature of a set E, $\overline{C_k}^f(E)$ $(0 \leq k \leq n)$ is defined as

$$\lim_{\delta \to 0} \frac{1}{-\ln \delta} \int_\delta^1 \epsilon^{s_k} C_k(E(\epsilon)) \epsilon^{-1} d\epsilon$$

where $C_k(E(\epsilon))$ denotes the k^{th} total curvature of the ϵ-distance set $E(\epsilon)$ of E, s_k is given by

$$s_k = \inf\{t : \epsilon^t C_k^{var}(E(\epsilon)) \to 0 \text{ as } \epsilon \to 0\},$$

where $C_k^{var}(E(\epsilon))$ denotes the k^{th} total variation curvature of ϵ-distance set of E. In particular, when $k = n$, $C_k(S(\epsilon)) = |S(\epsilon)|$.

We would like to investigate the relation between sets of finite α-packing measure, α-coherent sets and sets for which n^{th} fractal curvature measure exists. Further, we would like to study the behaviour of the Fourier transform of the measures supported on these sets.

Bibliography

[1] S. Agmon and L. Hormander, *Asymptotic properties of solutions of differential equations with simple characteristics*, J. Analyse Math., 30(1976), 1-38.

[2] M. L. Agranovsky and E. K. Narayanan, *L^p-Integrability, Supports of Fourier Transforms and Uniqueness for Convolution Equations*, J. Fourier Anal. Appl., 10 (2004), no. 3, 315-324.

[3] J. G. Bak and A. Seeger, *Extensions of the Stein-Tomas theorem*, Mathematical Research Letter, (4)18 (2011), 767-781.

[4] C. Bennett, R. Sharpley *Interpolation of operators*, Pure and Applied Mathematics, 129. Academic Press, Inc., Boston, MA, 1988.

[5] A. S. Besicovitch, *On existence of subsets of finite measures of sets of infinite measure*, Indag. Math. 14 (1952) 339-34.

[6] A. Beurling, *On a closure problem*, Ark. Mat. 1, (1951). 301-303.

[7] C. D. Cutler, *The Density theorem and Hausdorff Inequality for Packing measure in General Metric Spaces*, Ill. J. Math., 39(1995), 676-694

[8] W. F. Donoghue, Jr., *Distributions and Fourier Transforms*, Pure and Applied Mathematics, Vol 32 (1969), Academic Press, New York and London.

[9] R. E. Edwards, *Spans of translates in $L^p(G)$*, J. Austral. Math. Soc. 5 (1965), 216-233.

[10] M. B. Erdogan, *A note on the Fourier transform of fractal measures*, Math. Res. Lett. 11 (2004), no. 2-3, 299-313.

[11] M. B. Erdogan and D. M. Oberlin, *Restricting Fourier transforms of measures to curves in* $\mathbb{R}^3$ Canad. Math. Bull. 56 (2013), no. 2, 326-336.

[12] K. J. Falconer, *The geometry of fractal sets*, Cambridge tracts in Mathematics (85), Cambridge University press(1985).

[13] H. Federer, *Geometric measure theory*, Springer-Verlag(1969).

[14] K. Hambrook and I. Laba, *On the sharpness of Mockenhaupt's restriction theorem*, Geom. Funct. Anal. 23(2013) 1262-1277

[15] C. S. Herz, *A note on the span of translations in* L^p, Proc. Amer. Math. Soc. 8 (1957), 724-727.

[16] L. Hormander, *The Analysis of Linear Partial Differential Operators*, Vol. I. Springer-Verlag(1983), Berlin.

[17] S. Hudson and M. Leckband, *Hardy's Inequality and Fractal Measures*, J. Functional Analysis, 108(1992), 133-160.

[18] J. E. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. J., 30(1981), 713-747.

[19] J. P. Kahane and R. Salem, *Ensembles Parfaits et Series Trigonometriques*, Hermann, 1963.

[20] M. Kinukawa, *A note on the closure of translations in* L^p, Töhoku Math. J. 18(1966), 225-231.

[21] K. S. Lau, *Fractal measures and mean* p*-Variations*, J. Functional Analysis, 108(1992), 427-457.

[22] K. Lau and J. Wang, *Mean quadratic variations and Fourier asymptotics of self-similar measures*, Mh. Math. 115(1993), 99-132.

[23] N. Lev and A. Olevskii, *Wiener's 'closure of translates' problem and Piatetski-Shapiro's uniqueness phenomenon*, Ann. of Math. (2) 174 (2011), no. 1, 519-541.

[24] P. Mattila, *Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability*, Cambridge Studies in Advanced Mathematics, 44(1995), Cambridge University Press, Cambridge.

[25] P. Mattila, *Spherical averages of Fourier transforms of measures with finite energy; Dimension of intersections and Distance sets*, Mathematika, 34 (1987), 201-228.

[26] T. Mitsis, *A Stein-Tomas restriction theorem for general measures*, Puble. Nath. Debrecen, 60 (2002), 89-99.

[27] G. Mockenhaupt, *Salem sets and restriction properties of Fourier transforms*, Geom. Funct. Anal., 10 (2000), 1579-1587.

[28] E. K. Narayanan and R. Rawat, *L^p Wiener Tauberian theorems for $M(2)$*, Math. Z. 265 (2010), 437-449.

[29] D. J. Newman, *The closure of translates in L^p*, Amer. J. Math. 86 (1964), 651-667.

[30] H. Pollard, *The closure of translations in L^p*, Proc. Amer. Math. Soc. 2 (1951), 100-104.

[31] R. Rawat and A. Sitaram, *The injectivity of the Pompeiu transform and L^p-analogues of the Wiener Tauberian theorem*, Israel J. Math. 91 (1995), no. 1-3, 307-316.

[32] J. M. Rosenblatt and K. L. Shuman, *Cyclic functions in $L^p(\mathbb{R}),\ 1 \leq p < \infty$*,J. Fourier Anal. Appl. 9 (2003), 289-300.

[33] W. Rudin, *Functional analysis*, McGraw-Hill Series in Higher Mathematics, McGraw-Hill Book Co., New York-Dsseldorf-Johannesburg, 1973.

[34] R. Salem, *On singular monotonic functions whose spectrum has a given Hausdorff dimension*, Ark. Mat. 1, (1951). 353-365.

[35] A. Salli, *On the Minkowski dimension of strongly porous fractal sets in $\mathbb{R}^n$*, Proc. London Math. Soc. (3) 62 (1991),353-372.

[36] I. E. Segal, *The span of the translations of a function in a Lebesgue space*, Proc. Nat. Acad. Sci. U. S. A. 30 (1944), 165-169.

[37] K. S. Senthil Raani, *L^p-Integrability, Dimensions of Supports of Fourier transforms and applications*, J. Fourier Anal. Appl., 20 (2014), no. 4, 801-815.

[38] P. Sjolin, *Spherical harmonics and spherical averages of Fourier transforms*, Rend. Sem. Mat. Univ. Padova 108 (2002), 41-51.

[39] P. Sjolin and F. Soria, *Estimates of averages of Fourier transforms with respect to general measures*, Proc. Roy. Soc. Edinburgh Sect. A 133 (2003), no. 4, 943-950.

[40] E. M. Stein, *Harmonic Analysis - Real Variable Methods, Orthogonality and Oscillatory Integrals*, Princeton University Press, Princeton, New Jersey (1993).

[41] R. S. Strichartz, *Fourier asymptotics of fractal measures*, J. Functional Analysis, 89(1990), 154-187.

[42] M. Sugiura, *Unitary Representations and Harmonic Analysis, An Introduction*, Kodansha Scientific books, Tokyo (1975).

[43] C. Tricot, *Two definitions of fractional dimension*, Math. Proc. Cambridge Philos. Soc. 91 (1982), 57-74.

[44] N. Wiener, *Tauberian theorems*, Ann. of Math. 33 (1932), 1.

[45] S. Winter, *Curvature measures and fractals*, Dissertationes Math. (Rozprawy Mat.) 453 (2008), 66.

[46] T. Wolff, *Decay of circular means of Fourier transform of measures*, Internat. Math. Res. Notices 1999, no. 10, 547-567.